ON THE REGULAR SET OF BMO SOLUTIONS TO STRONGLY COUPLED ELLIPTIC SYSTEM

Dung Le

Miami Dec 2012

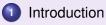
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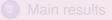
Regularity Set

December 12, 2012 1 / 20

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Function spaces and Assumptions

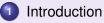




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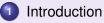
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December 12, 2012 2 / 20

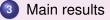


- 2 Function spaces and Assumptions
 - 3 Main results
 - A Sketch of proof

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2 Function spaces and Assumptions

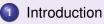


4 Sketch of proof

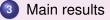
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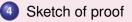
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December 12, 2012 2 / 20



Punction spaces and Assumptions





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Regularity Set

December 12, 2012 2 / 20

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Here:

- **(**) *Q* is a bounded domain in \mathbb{R}^n (n > 1),
- 3 u, f have vector valued in \mathbb{R}^m (m > 1) (for simplicity f is bounded),
- If A(u, Du) is a matrix $n \times m$.

We will characterize the regular set of a BMO weak solution u. The matrix A(u, Du) can be nonregular and singular.

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$$\begin{cases} -\operatorname{div}(A(u, Du)) = f & \text{in } Q, \\ u = 0 & \text{on } \partial Q \end{cases}$$
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- Counterexamples exist for systems: bounded weak solutions may not be Hölder continuous.
- Partial regularity: bounded weak solutions for regular elliptic systems are Hölder continuous on a full measure and open set.
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Function spaces

a locally integrable vector valued function $u : Q \to \mathbb{R}^m$ is BMO if the seminorm

$$\|u\|_{BMO(Q)} = \sup_{B \subset Q} \int_{B} |u - u_B| \, dz < \infty,$$

where the supremum is taken over all balls $B \subset Q$.

Let μ be a doubling measure on \mathbb{R}^n and Ψ be a μ -measurable nonnegative function and $\gamma > 1$. We say that $\Psi \in A_{\gamma}(\mu)$ or Ψ is an $A_{\gamma}(d\mu)$ weight if the quantity

$$[\Psi]_{\gamma} = \sup_{B} \left(\int_{B} \Psi \, d\mu \right) \left(\int_{B} \Psi^{1-\gamma'} \, d\mu \right)^{\gamma-1} < \infty.$$
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Here, $\gamma' = \gamma/(\gamma - 1)$ and the supremum is taken over all balls *B* of \mathbb{R}^n . The $A_{\infty}(\mu)$ class is defined by $A_{\infty}(\mu) = \cup_{\gamma > 1} A_{\gamma}(\mu)$.

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Uniform ellipticity and Continuity

A.1) (Ellipticity) For any $u \in \mathbb{R}^m$ and $\zeta, \xi \in \mathbb{R}^{nm}$ there are nonnegative constants $\lambda(u), \Lambda(u)$ and p > 1 such that

$$\lambda(u)|\zeta|^{\rho} \leq \langle A(u,\zeta),\zeta\rangle \leq \Lambda(u)|\zeta|^{\rho}.$$
(3)

and

$$\langle A(u,\zeta) - A(u,\xi), \zeta - \xi \rangle \ge \lambda(u) |\zeta - \xi|^{p}.$$
 (4)

Moreover, there are nonnegative constants λ_0,λ_1 such that

$$\lambda(\boldsymbol{u}) \ge \lambda_0, \tag{5}$$

$$\Lambda(u) \le \lambda_1 \lambda(u). \tag{6}$$

A.2) (Continuity) For any $u, v \in \mathbb{R}^m$ and $\zeta \in \mathbb{R}^{nm}$ there is a function $\Delta(u, v)$ such that

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On the ellipticity constant

L.1) $\lambda(u)$ is quasi convex in the sense that there is a constant *C* such that for any vector valued function *u* and any ball $Q_R \subset \mathbb{R}^n$

 $\lambda(u_R) \le C(\lambda(u))_R. \tag{8}$

L.2) $A(u, \zeta)$ is Hölder continuous in u. In fact, for the ellipticity and continuity constants $\lambda(u), \Delta$ in A.2) there are positive $\alpha, \beta, \theta_1, \alpha_0, \beta_0, \theta_0$ such that for any vectors $u, v \in \mathbb{R}^m$

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and

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Higher integrability result with weights

Using only the assumption that $\lambda(u)$ belongs to some $A_q(dz)$ class, with q being determined by p, n we will show that there exist positive numbers γ_*, C depending only on the A_q characterization of $\lambda(u)$ such that for any $\gamma \in (1, \gamma_*]$ and $\tau \in (0, 1)$ the following higher integrability result holds

$$\left(\oint_{Q_{\tau R}} |Du|^{\gamma p} d\mu\right)^{\frac{1}{\gamma p}} \leq C \left(\oint_{Q_{R}} |Du|^{p} d\mu\right)^{\frac{1}{p}}, \quad d\mu = \lambda(u) dz. \quad (11)$$

On the parameters

With γ_* being described earlier, we will consider

P.1) For some $\gamma \in (1, \gamma_*]$

$$\alpha > \frac{1}{p'}, \quad \beta \le \frac{1}{p} - \frac{1}{p'\gamma'}, \quad \alpha \le \beta, \tag{12}$$

$$\beta_0 > \alpha_0, \quad \beta_0 \le 1/\gamma'. \tag{13}$$

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P.3) There is $\gamma \in (1, \gamma_*]$ such that $\alpha + \beta - 1 < \min\{\frac{1}{p'\gamma'}, p\theta_1\}$ and $\alpha_0 + \beta_0 - 1 \leq \min\{\frac{1}{\gamma'_2}, \theta_0\}$

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Le (UTSA) Regularity Set December 12, 2012

On the regular set of a BMO weak solution u to (1)

Theorem

Assume A.1), A.2), B), L.1), L.2), P.1) or P.2), and P.3). Assume also that $\lambda_0 > 0$. There exist positive constants C, ε_0, ν_0 depending only on $\|u\|_{BMO}, \|\lambda(u)\|_{BMO}$ such that for

$$\Sigma_0 = \{ z_0 \in Q : \liminf_{R \to 0} - \int_{Q_R(z_0)} |u - u_R|^p \lambda(u) \, dz < \varepsilon_0 \}$$
(16)

then for any balls Q_{ρ}, Q_{R} contained in Q and centered at $z_{0} \in \Sigma_{0}$ the following holds

$$\int_{Q_{\rho}} |u - u_{\rho}|^{\rho} \lambda(u) \, dz \le G\left(\frac{\rho}{R}\right)^{\nu_{0}} - \int_{Q_{R}} |u - u_{R}|^{\rho} \lambda(u) \, dz.$$
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Moreover u is locally Hölder continuous in Σ_0 .

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Quasi concave $\lambda(u)$

Our next result concerns the case where $\lambda(u)$ does not satisfy the quasi convexity condition L.1) but

L.1') $\lambda(u)$ is quasi concave in the sense that there is a constant *C* such that for any vector valued function *u* and any ball $Q_R \subset \mathbb{R}^n$

$$\lambda(u_R) \ge C(\lambda(u))_R. \tag{18}$$

In this case, we will assume that u locally has the vanishing mean oscillation (VMO) property. We say that a locally integrable function u has VMO property at a point z_0 , or VMO at that point, if

$$\lim_{R\to 0} \frac{1}{f_{Q_{z_0,R}}} |u - u_{Q_{z_0,R}}| \, dz = 0.$$

December 12, 2012 11 / 20

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December 12, 2012 11 / 20

Second theorem

We have the following result.

Theorem

holderthm2 Assume that A.1), A.2), B), L.1') hold, $\lambda_0 > 0$, and that there are positive α, β, θ_1 such that for any vectors $u, v \in \mathbb{R}^m$

$$|\Delta(u, v)| \le \max\{\lambda(u)^{\alpha}\lambda(v)^{\beta}, \lambda(v)^{\alpha}\lambda(u)^{\beta}\}|u-v|^{\theta_{1}},$$
(19)

with $\alpha + \beta \leq 1/p$. Then a BMO weak solution u to (1) is Hölder at z_0 if u and $\lambda(u)$ have VMO property at z_0 .

Of course, if $\lambda(u)$ is Hölder in u then $\|\lambda(u)\|_{BMO(Q_R)}$ is bounded by $\|u\|_{BMO(Q_R)}$, and thus the above theorem asserts that u is Hölder at a point if and only if it is VMO there.

The singular case

We can drop the assumption that $\lambda(u)$ is BMO in B) to have the following result concerning the singular case $\lambda_0 = 0$.

We only assume A.1), A.2), L.1), L.2) with $\lambda_0 =$ 0. In addition, suppose that P.1)-P.3) are verified and

$$\alpha + \beta - 1 \text{ and } \alpha_0 + \beta_0 - 1 > 0,$$
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$$\lambda(u) \in A_{\gamma_0} \text{ and } \lambda(u)^{-1} \in A_{\sigma_0}$$
 (21)

 $\gamma_0 \in (1, p(n-1)/n] \text{ and } \sigma_0 \in (1, \min\{2, n/(n-\alpha(p))\}).$

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The third theorem

Theorem

There are constants $C, \varepsilon_0 \nu_0$ depending on $\gamma_0, \gamma_0, [\lambda(u)]_{\gamma_0}$ and $[\lambda(u)^{-1}]_{\sigma_0}$ such that if we define

$$\Sigma_0 = \{ z_0 \in Q : \liminf_{R \to 0} \frac{1}{\int_{Q_R(z_0)}} |u - u_R|^p \lambda(u) \, dz < \varepsilon_0 \}$$
(22)

then for any balls Q_{ρ}, Q_R in Q and centered at $z_0 \in \Sigma_0$ the following decay estimate holds

$$\int_{Q_{\rho}} |u - u_{\rho}|^{p} \lambda(u) \, dz \leq C \left(\frac{\rho}{R}\right)^{\nu_{0}} \int_{Q_{R}} |u - u_{R}|^{p} \lambda(u) \, dz.$$
(23)

Notes

Concerning the Hölder continuity of *u* we consider a point $z_0 \in \Sigma_0$ where $\lambda(u(z_0)) > 0$. Therefore, if $q \in (1, p)$ then

$$\int_{\mathcal{Q}_{\rho}} |u-u_{\rho}|^{q} dz \leq \left(\int_{\mathcal{Q}_{\rho}} (\lambda(u))^{-\frac{q}{p-q}} dz \right)^{\frac{p-q}{p}} \left(\int_{\mathcal{Q}_{\rho}} |u-u_{\rho}|^{p} \lambda(u) dz \right)^{\frac{q}{p}}$$

By Lebesgue's theorem and the fact that $\lambda(u(z_0)) > 0$, the first factor on the right is bounded when ρ is sufficiently small. Combining these facts with (23) we assert that *u* is Hölder continuous at z_0 .

Dung Le (UTSA)

Regularity Set

December 12, 2012 15 / 20

Tools

Caccioppoli-type inequality:

$$\int_{Q_R} |Du|^p \lambda(u) \, dz \leq CR^{-p} \int_{Q_{2R}} |u - \int_{Q_{2R}} u \, dz|^p \lambda(u) \, dz.$$
(24)

Weighted Sobolev-Poincaré inequality: Let $d\mu = \lambda(u)dz$ and $l = p\gamma n/(\gamma n + p)$. Then

$$-\int_{Q} |u - -\int_{Q} u \, dz|^{p} \, d\mu \leq C([\lambda(u)]_{\gamma})R^{p} \left(-\int_{Q} |Du|^{\prime} \, d\mu\right)^{\frac{p}{\gamma}}.$$
 (25)

Higher integrability of gradients: There are $\gamma \in (1, \gamma_*)$ and $\tau \in (0, 1)$ such that

$$\left(\int_{Q_{\tau R}} |Du|^{\gamma p} d\mu\right)^{\frac{1}{\gamma p}} \leq C(\gamma_*, [\lambda(u)]_{\gamma_0}, \tau) \left(\int_{Q_R} |Du|^p d\mu\right)^{\frac{1}{p}}.$$
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Here, ${\it C}(\gamma_*, [\lambda({\it u})]_{\gamma_0}, au)$ is bounded in au^{-1}

Dung Le (UTSA)

Regularity Set

December 12, 2012 16 / 20

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Regularity Set

December 12, 2012 16 / 20

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Approximation

For any ball Q_R contained in Q we will compare our solution u with that of

$$\begin{cases} -\operatorname{div}(A(u_R, DV)) = f & \text{in } Q_R, \\ V = u & \text{on } \partial Q_R. \end{cases}$$
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Splitting domains: $\lambda(u), \lambda(u_R)$ are NOT comparable. Let Q_R^1 be any measurable subset of Q_R and $Q_R^2 = Q_R \setminus Q_R^1$. For w = V - u, we have

$$\int_{Q_R^1} \lambda(u_R) |Dw|^p \, dz + \int_{Q_R^2} \lambda(u) |Dw|^p \, dz \leq \int_{Q_R^1} |\Delta(u, u_R)| |Du|^{p-1} |Dw| \, dz + \int_{Q_R^2} |\Delta(u, u_R)| |DV|^{p-1} |Dw| \, dz.$$

Dung Le (UTSA)

Regularity Set

December 12, 2012 17 / 20

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$$\begin{split} \int_{Q_R^1} \lambda(u_R) |Dw|^p \, dz + \int_{Q_R^2} \lambda(u) |Dw|^p \, dz \leq \\ \int_{Q_R^1} |\Delta(u, u_R)| |Du|^{p-1} |Dw| \, dz + \int_{Q_R^2} |\Delta(u, u_R)| |DV|^{p-1} |Dw| \, dz. \end{split}$$

December 12, 2012 17 / 20

Decay estimate for DV

For some constant $\alpha(p) > 0$

$$f_{Q_{\tau R}} |DV|^{p} dz \leq C\tau^{-p+\alpha(p)} f_{Q_{R}} |DV|^{p} dz.$$
(28)

This usually gives Hölder continuity for *V*. Can this be *transfered* to *Du*?

Decay estimate for *DV*

For some constant $\alpha(p) > 0$

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This usually gives Hölder continuity for *V*. Can this be *transfered* to *Du*?

Decay estimate for *Du*

For any $\tau \in (0, 1/2)$ and positive K_R such that $K_R \leq C_0 \min\{(\lambda(u))_R, \lambda(u_R)\}$, we can find positive constants C_*, ν such that

$$(\tau R)^{p} \oint_{Q_{\tau R}} |Du|^{p} \lambda(u) dz \leq C_{*} [\tau^{-n+p} \Phi(u, R) + \tau^{\nu}] R^{p} \oint_{Q_{2R}} |Du|^{p} \lambda(u) dz.$$
(29)

Here, ν depends on n, p and C_* depends on $n, p, C_0, [\lambda(u)]_{\gamma_0}, [\lambda(u)]_{\gamma'}$ and $[\lambda(u)^{-1}]_{\sigma_0}$. Splitting Q_R we can prove that $\Phi(u, R)$ is small if the *weighted mean* oscillation of u is. Decay estimate for Du then follows

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Decay estimate for *Du* then follows.

$\Phi(u, R)$

For any measurable subset
$$Q_R^1$$
 of Q_R and $Q_R^2 = Q_R \setminus Q_R^1$
 $\Phi(u, R) = \Phi_0(u, R) + \Phi_1(u, R) + \Phi_2(u, R)$
 $\Phi_0(u, R) = \left(\frac{1}{|Q_R|} \int_{Q_R^1} \left| \frac{\lambda(u) - K_R}{\lambda(u)^{\frac{1}{\gamma}} (\lambda(u))_R^{\frac{1}{\gamma'}}} \right|^{\gamma'} dz \right)^{\frac{1}{\gamma'}},$ (30)
 $\Phi_1(u, R) = \left(\frac{1}{|Q_R|} \int_{Q_R^1} \left| \frac{|\Delta(u, u_R)|}{\lambda(u_R)^{\frac{1}{p}} \lambda(u)^{\frac{1}{p'\gamma'}}} \right|^{p'\gamma'} dz \right)^{\frac{1}{\gamma'}} (\lambda(u))_R^{-\frac{1}{\gamma}},$ (31)

and

$$\Phi_{2}(u,R) = \left(\frac{1}{|Q_{R}|} \int_{Q_{R}^{2}} \left|\frac{|\Delta(u,u_{R})|}{\lambda(u)^{\frac{1}{p}}}\right|^{p'\gamma'} dz\right)^{\frac{1}{\gamma'}} (\lambda(u))_{R}^{-1}.$$
 (32)

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December 12, 2012 20 / 20