Global and local averaging for integrodifferential equations

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Motivation

Reaction-diffusion equation for individuals or genes

$$u_t = Du_{xx} + F(u)$$

- Infinite homogeneous landscape: Invasion speed $c^* = 2\sqrt{DF'(0)}$ Fisher (1937), Weinberger (1982)
- Single patch: Minimal domain size $L^* = \pi\sqrt{D/F'(0)}$ Skellam (1951), Kierstadt and Slobotkin (1953)
- Many patches?

Heterogeneous landscapes

Reaction-diffusion equation in a periodic environment

$$u_t = (D(x)u_x)_x + F(u, x)$$

with D and $F(u, \cdot)$ of the same period.

Persistence conditions and invasion speeds

- Exact conditions for piecewise constant landscapes

  Shigesada et al (1986)

- Abstract results for periodic landscapes

  Weinberger (2002), Berestycki et al (2005)

Homogenization

Assume: Small and large spatial scale
Average over the small scale

$$u_t = \widetilde{D} u_{xx} + \widetilde{F}(u).$$

where

$$\widetilde{D} = \left(\frac{1}{L} \int_0^L \frac{dy}{D(y)} \right)^{-1}, \qquad \widetilde{F}(u) = \frac{1}{L} \int_0^L F(u, y) dy,$$

- $\widetilde{F}'(0) > 0 \Rightarrow$ persistence
- $D = 0$ somewhere $\Rightarrow$ no spread
- No correlations between D and F enter the equation.
- Requires movement, not applicable to sedentary stages.

Outline

1. Non-local dispersal
2. Persistence via Global Averaging
3. Persistence via Local Averaging
4. Invasion speeds via Global Averaging

Outline

Modeling movement

- Random walk on real line
- Time between moves has Poisson distribution with mean μ
- Move length distribution K

$$
\begin{aligned}
u_t(t,x) &= -\mu u(t,x) + \int_{-\infty}^{\infty} K(x-y)\mu u(t,y)dy \\
&= -\mu u(t,x) + [K * (\mu u)](t,x)
\end{aligned}
$$

- K is symmetric and exponentially bounded
- Moment generating function $M(s) = \int k(x)e^{sx}dx$

Including population dynamics

1. Continuous movement and reproduction (linear)

$$u_t = (b - m)u - \mu u + K * (\mu u).$$

   b: birth rate, m: mortality rate

2. Mobile offspring, sessile adults (linear)

$$u_t = -mu + \gamma K * (bu)$$

   γ: probability of successful establishment

3. Distributed contacts Mollison (1991)

$$I_t = \beta(N - I)(K * I) - \alpha I,$$

4. Distributed infectives Medlock and Kot (2003)

$$I_t = \beta(N - I)I - \alpha I + \mu(K * I - I)$$

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The linear model in a heterogeneous landscape

We consider the model

$$u_t(t,x) = f(x)u(t,x) + g(x)[K * (hu)](x)$$

with $0 \le g \le 1, h = h(x) \ge 0$.

Periodic Landscape:

- Patch type i of length L_i
- Period $L_1 + L_2 = L$ and fraction $p = L_1/L$
- Parameter functions piecewise constant: $f(x) = f_i$ on patch i
- Dispersal kernel origin dependent: $K_i(z)$ on patch i

Outline

Averaging I

Persistence condition

$$\lambda u(x) = f(x)u(x) + g(x) \int_{-\infty}^{\infty} K(x-y;y)h(y)u(y)dy$$

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Scale space $x = Lz, y = Lw$, $\tilde{u}(z) = u(x/L)$

$$\lambda\tilde{u}(z) = \tilde{f}(z)\tilde{u}(z) + \tilde{g}(z)\int_{-\infty}^{\infty} L\widetilde{K}(L(z-w);w)\tilde{h}(w)\tilde{u}(w)dw$$

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Periodicity

$$\lambda \tilde{u}(z) = \tilde{f}(z)\tilde{u}(z) + \tilde{g}(z)\int_{0}^{1}\sum_{n} L\widetilde{K}(L(z-w-n);w)\tilde{h}(w)\tilde{u}(w)dw.$$

Averaging II

Riemann sum

$$\lim_{L\to 0}\sum_n L\widetilde{K}(L(z-w-n);w)=\int_{-\infty}^{\infty}K(v;w)dv=1.$$

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Globally averaged equation

$$\lambda\tilde{u}(z)=\tilde{f}(z)\tilde{u}(z)+\tilde{g}(z)\int_0^1\tilde{h}(w)\tilde{u}(w)dw,$$

Averaging II

Riemann sum

$$\lim_{L \to 0} \sum_n L\widetilde{K}(L(z-w-n); w) = \int_{-\infty}^{\infty} K(v; w) dv = 1.$$

Globally averaged equation

$$\lambda \tilde{u}(z) = \tilde{f}(z)\tilde{u}(z) + \tilde{g}(z) \int_0^1 \tilde{h}(w)\tilde{u}(w) dw,$$

If only h depends on x:

$$\lambda = f + g \int_0^1 \tilde{h}(w) dw$$

Dispersal, via K, helps to average local growth. But f, g outside the integral have to be constant.

Averaging III

Eigenvalue equation after scaling and limit

$$\lambda \tilde{u}(z) = \tilde{f}(z)\tilde{u}(z) + \tilde{g}(z) \int_0^1 \tilde{h}(w)\tilde{u}(w)dw,$$

Averaging III

Eigenvalue equation after scaling and limit

$$\lambda\tilde{u}(z) = \tilde{f}(z)\tilde{u}(z) + \tilde{g}(z)\int_0^1 \tilde{h}(w)\tilde{u}(w)dw,$$

Piecewise constant parameter functions

$$\lambda u_1(z) = f_1 u_1(z) + g_1 h_1 \int_0^p u_1(w)dw + g_1 h_2 \int_p^1 u_2(w)dw,$$
$$\lambda u_2(z) = f_2 u_2(z) + g_2 h_1 \int_0^p u_1(w)dw + g_2 h_2 \int_p^1 u_2(w)dw,$$

System of two equations: u_i is density on patch i

Averaging Result

Consider the eigenvalue problem

$$\lambda u(x) = f(x)u(x) + g(x)\int_{-\infty}^{\infty} K(x-y;y)h(y)u(y)dy$$

with piecewise constant, L-periodic coefficient functions.

In the limit $L \to 0$, the dominant eigenvalue satisfies

$$\lambda \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} f_1 + g_1 h_1 p & g_1 h_2(1-p) \\ g_2 h_1 p & f_2 + g_2 h_2(1-p) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Example

Continuous movement and growth

$$u_t = (b - m)u - \mu u + K * (\mu u)$$

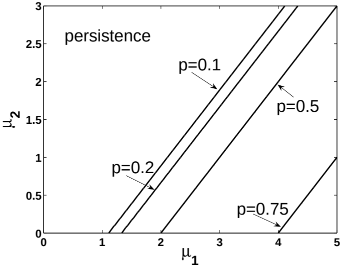
b: birth rate, m: mortality rate, $r = b - m$ net growth

Scale $r_1 = 1, r_2 < 0$.
Persistence condition

$$\mu_1 < \frac{1}{1-p}$$

or

$$|r_2| < \frac{\mu_2 p}{\mu_1(1-p) - 1}.$$



Persistence boundary for zero mean growth

Outline

Patch scale - idea

Eigenvalue equation as before

$$\lambda\tilde{u}(z) = \tilde{f}(z)\tilde{u}(z) + \tilde{g}(z)\int_0^1 \widehat{K}(L(z-w-n);w)\tilde{h}(w)\tilde{u}(w)dw.$$

Now split into patch types

$$\lambda u_1(z) = f_1 u_1(z) + g_1 h_1 \int_0^p \widehat{K}_1(z-w)u_1(w)dw + g_1 h_2 \int_p^1 \widehat{K}_2(z-w)u_2(w)dw,$$
$$\lambda u_2(z) = f_2 u_2(z) + g_2 h_1 \int_0^p \widehat{K}_1(z-w)u_1(w)dw + g_2 h_2 \int_p^1 \widehat{K}_2(z-w)u_2(w)dw,$$

Take averages $\bar{u}_1 = \frac{1}{p}\int_0^p u_1(z)dz$, and $\bar{u}_2$ accordingly.

Patch scale - result

To lowest order, the averages satisfy

$$\lambda \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix} = \begin{bmatrix} f_1 + g_1 h_1 s_1^{11} & g_1 h_2 \frac{1-p}{p} s_2^{12} \\ g_2 h_1 \frac{p}{1-p} s_1^{21} & f_2 + g_2 h_2 s_2^{22} \end{bmatrix} \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{bmatrix}.$$

Where the average dispersal success from patch j to patch i is

$$s^{ij} = \frac{1}{|\Omega_j|} \int_{\Omega_j} \int_{\Omega_i} K(x-y) dx dy.$$

Global averaging results when s^{ij} equals the fraction of type i patches.

The s^{ij}-terms contain local movement information

Example

Continuous movement and growth

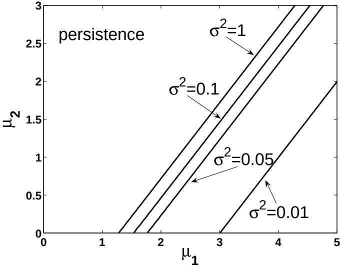
$$u_t = (b - m)u - \mu u + K * (\mu u)$$

b: birth rate, m: mortality rate, $r = b - m$ net growth

Laplace Kernel

$$K(x) = \frac{1}{2d} \exp(-|x|/d)$$

variance $\sigma^2 = 2d^2$



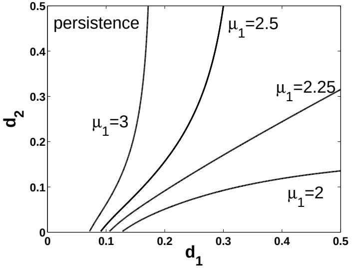
Persistence boundary for zero mean growth, $p = 0.2$

Example - continued

Persistence boundary from dispersal distances

d_i: dispersal distance from patch i

$\mu_2 = 1$
$p = 0.2$



No persistence for global averaging

Outline

Traveling periodic wave - linear

Ansatz

$$u(t,x) = e^{-s(x-ct)}v(x)$$

Eigenvalue equation

$$scv(x) = f(x)v(x) + g(x)\int_{-\infty}^{\infty} K(x-y;y)e^{s(x-y)}h(y)v(y)dy$$

As before (global averaging)

$$scv(z) = f(z)v(z) + g(z)\int_0^1 M(s;w)h(w)v(w)dw.$$

$$M(s;w) = \int_{-\infty}^{\infty} \tilde{K}(z';w)e^{sz'}dz'$$

Minimal speed

Model equation

$$u_t(t,x) = f(x)u(t,x) + g(x)[K * (hu)](x)$$

Result: In the fine-grain limit $L \to 0$, the minimal TW speed is

$$c = \min_{s>0} \frac{1}{s}\rho(s),$$

where $\rho(s)$ is the dominant eigenvalue of

$$\begin{bmatrix} f_1 + g_1 h_1 M_1(s) p & g_1 h_2 M_2(s)(1-p) \\ g_2 h_1 M_1(s) p & f_2 + g_2 h_2 M_2(s)(1-p) \end{bmatrix}$$

Example

Continuous movement and growth

$$u_t = (b - m)u - \mu u + K * (\mu u)$$

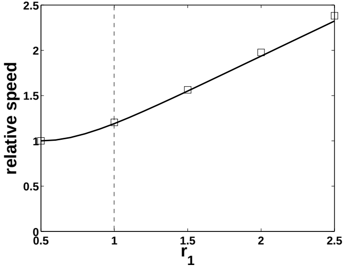
b: birth rate, m: mortality rate, $r = b - m$ net growth

Laplace Kernel (mean d)

$$K(x) = \frac{1}{2d} \exp(-|x|/d)$$

Moment generating function

$$M(s) = \frac{1}{1 - d^2 s^2}$$

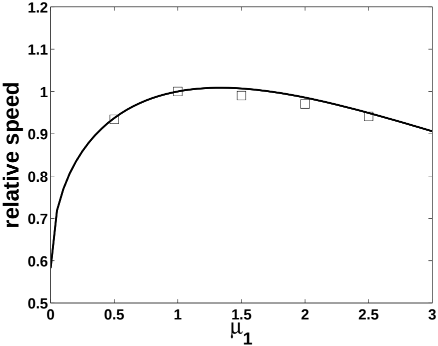


Speed with constant mean growth. Only r varies.

Example - continued

$r_1 = 1, r_2 = 0$

$\mu_2 = 1$



Maximum Speed for intermediate movement rate μ_1.

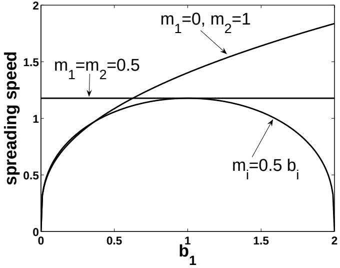
Example II

Mobile offspring, sessile adults (linear)

$$u_t = -mu + \gamma K * (bu)$$

γ: probability of successful establishment

constant average net growth
$p(b_1 - m_1) + (1-p)(b_2 - m_2) = 1/2$



Speeds for different scenarios

Conclusions

1. Technique of homogenization
    - deal with integrals
    - deal with non-mobile compartment
    - patch averaging retains movement information
2. Results for specific models
    - Not just averaged growth and dispersal
    - Correlations matter
3. Extensions
    - Apply to kernels with movement behavior (Jeff Musgrave)
    - patch averaging for RDE (with Christina Cobbold)
    - Apply to reaction-diffusion equations with no-mobile stage

F. Lutscher (2010) Nonlocal dispersal and averaging in heterogeneous landscapes
Applicable Analysis 89(7): 1091–1108