

Using Optimal Control of Parabolic PDEs to Investigate Population Questions

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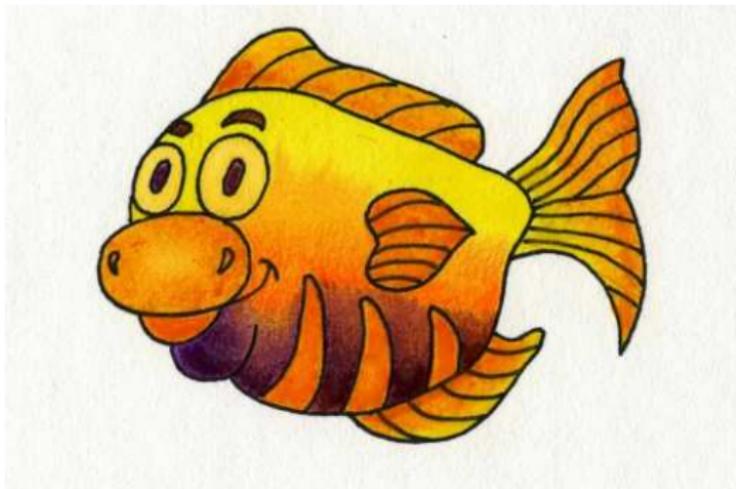
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dedicated to Chris Cosner

- Motivation
- Fish Harvesting Examples
(collaborators: Neubert, Herrera, Joshi)
- Example with Control on Advection Direction
(collaborators: Phan, Finotti, Y. Lou, Ding, Ye)
- Conclusions and Discussion.

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Harvesting



Choosing Movement Direction



Motivation

- No-take marine reserves may be a part of optimal harvest strategy designed to maximize yield.
- Marine reserves can protect habitat and defend endangered stock from overexploitation.
- Marine reserves as a part of fishery management plan are controversial.

Neubert (Ecology Letters, 2003)

Examined steady state of this population PDE

$$N_t = DN_{xx} + rN \left(1 - \frac{N}{K}\right) - qE(x)N$$

Steady state

$$0 = DN_{xx} + rN \left(1 - \frac{N}{K}\right) - qE(x)N, 0 < x < L.$$

- rescaled equation $u'' + u(1 - u) - h(x)u = 0$
- $u = 0$ at the boundary $x = 0$ and $x = L$
- max yield $\int_0^L h(x)u(x)dx$
- **u STATE and h CONTROL**
- **TOOL: Pontryagin's Maximum Principle**
- Depending on length of domain, marine reserves are part of optimal harvesting strategy.
- For large length, there are many intervals of no harvest(reserve), leading to 'chattering'.
For small length, there is one reserve in the middle.

Idea in 1D

Rough Idea in 1 dimensional domain:



Optimal control and Pontryagin's Maximum Principle

Pontryagin and his collaborators developed optimal control theory for ordinary differential equations about 1950.

Pontryagin's KEY idea was the introduction of the adjoint variables to attach the differential equations to the objective functional (like a Lagrange multiplier attaching a constraint to a pointwise optimization of a function).

Converted problem of finding an optimal control to maximize the objective functional subject to dynamic equations (with initial conditions) to maximizing the Hamiltonian pointwise.

To consider extensions... consider non-steady state, include time and more than 1 space variable. **Parabolic PDE**

There is no complete generalization of Pontryagin's Maximum Principle to PDEs.

After setting up a PDE with a control in a specified set and an objective functional, proving existence of an optimal control is a first step.

Necessary Conditions

To derive the necessary conditions, we need to differentiate the map

control \rightarrow objective functional

Note that the state contributes to the objective functional, so we also must differentiate the map

control \rightarrow state

The “sensitivity” is the derivative of the control-to-state map. The sensitivity solves a PDE, which is linearized version of the state PDE.

How to find and use the adjoint function

The formal **adjoint** of the operator in the sensitivity PDE is found.

Transversality Condition: final time condition $\lambda = 0$ at $t = T$

nonhomogeneous term

$$\frac{\partial \text{integrand of } J}{\partial \text{state}}$$

Differentiate the objective functional $J(\text{control})$ with respect to the control.

Use the adjoint problem and the sensitivity problem to simplify and obtain the explicit characterization of an optimal control.

This work extends the work of Neubert

- includes both time and space
- multi-dimensional spatial domain
- non-steady state

Investigate the presence of marine reserves in optimal harvesting strategy.

We have completed the analysis for general semilinear parabolic PDE in a multidimensional domain but here we present a simpler case.

Parabolic Fishery Model

Our fishery model in domain $Q = \Omega \times (0, T)$ with $\Omega \subset \mathbb{R}^n$ is :

$$u_t = \Delta u + u(1 - u) - hu \quad \text{in } Q \quad (1)$$

with initial and boundary conditions:

$$\begin{aligned} u(x, 0) &= u_0(x) && \text{for } x \in \Omega \\ u(x, t) &= 0 && \text{on } \partial\Omega \times (0, T) \end{aligned}$$

- u represents fish population STATE
- h represents harvest rate CONTROL

We seek to maximize the objective functional over $h \in U$:

$$J(h) = \int_0^T \int_{\Omega} e^{-\delta t} h u \, dx \, dt \quad (2)$$

where $U = \{h \in L^\infty(Q) : 0 \leq h(x, t) \leq M \leq 1\}$ is class of admissible controls and $e^{-\delta t}$ represents a discount factor with interest rate δ .

$$(1 + \delta)/2 < M$$

This problem is linear in the control.

Existence of an Optimal Control

Solution space u in $V = L^2(O, T, H_0^1(\Omega))$ with
 u_t in $L^2(0, T; H^{-1}(\Omega))$

Note $u > 0$ in Q .

Theorem

There exists an optimal control h^ maximizing the functional $J(h)$ over U .*

Proof.

- Choose a maximizing sequence $\{h^n\}$ in U .
- Use a priori estimates.
- Use weak convergence results.



Theorem

The mapping $h \rightarrow u = u(h)$ is differentiable in the following sense:

$$\frac{u(h + \epsilon l) - u(h)}{\epsilon} \rightarrow \psi$$

weakly in V as $\epsilon \rightarrow 0$ for any $h \in U$ and $l \in L^\infty(Q)$ s.t. $(h + \epsilon l) \in U$ for ϵ small. The sensitivity ψ satisfies:

$$\begin{aligned} \psi_t &= \Delta \psi + \psi - 2u\psi - h\psi - lu && \text{in } Q \\ \psi(x, 0) &= 0 && \text{for } x \in \Omega \\ \psi(x, t) &= 0 && \text{on } \partial\Omega \times (0, T) \end{aligned} \quad (3)$$

Characterization of Opt Control

Theorem

Given an optimal control h^ and corresponding solution $u^* = u(h^*)$ there exists a weak solution $\lambda \in V$ with $\lambda_t \in L^2(0, T; H^{-1}(\Omega))$ satisfying the adjoint equation:*

$$\begin{aligned} -\lambda_t - \Delta\lambda - \lambda + 2u^*\lambda + h^*\lambda + \delta\lambda &= h^* \quad \text{in } Q \\ \lambda(x, t) &= 0 \quad \text{on } \partial\Omega \times (0, T). \end{aligned} \tag{4}$$

and transversality condition

$$\lambda(x, T) = 0 \quad \text{for } x \in \Omega.$$

Adjoint PDE, backwards with FINAL time condition

And furthermore the characterization of an OC:

$$h^*(x, t) = \begin{cases} 0 & \text{if } \lambda(x, t) > 1 \\ \frac{1+\delta}{2} & \text{if } \lambda(x, t) = 1 \\ M & \text{if } \lambda(x, t) < 1 \end{cases} \quad (5)$$

Note that possible bang-bang or singular cases.

Solve numerically the state and adjoint equations coupled with this optimal control characterization.

Optimal Control for Unexploited Stock Initial Condition

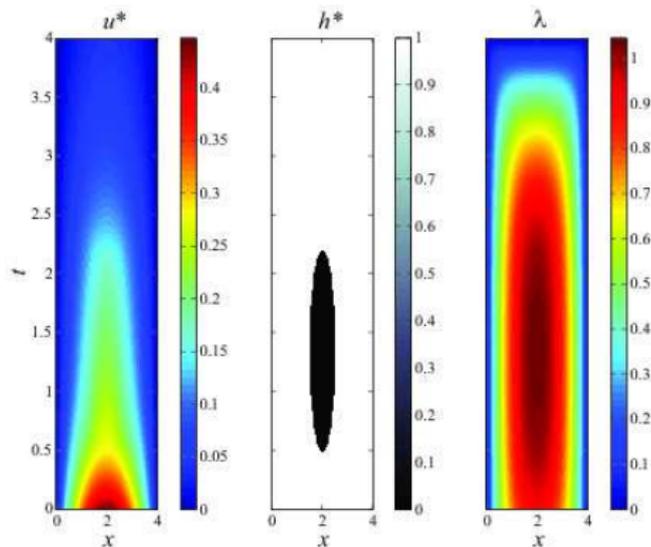
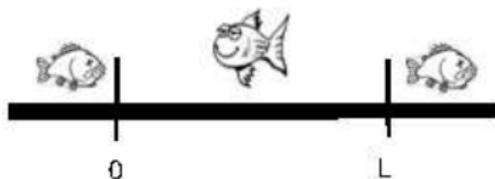
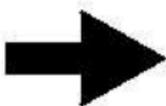
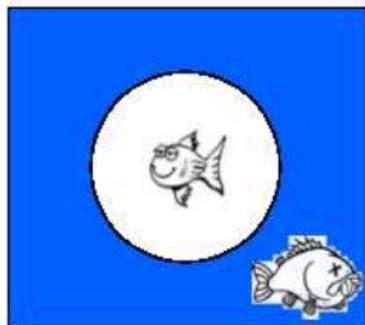


Figure: Final time 4, length of domain 4, discount .2

Here we obtain a marine reserve in the interior of the domain from optimal control results, while maximizing yield!

This work was generalized to include a more general parabolic operator with advection terms.

Dirichlet Boundary Condition



Effect of Boundary Conditions

NOTE that the Dirichlet boundary condition gives a type of heterogeneity to the problem.

If you use Neumann BC for this problem, the optimal control is constant and the optimal state is also constant.

What about Robin BC?

My student, Mike Kelly, is working on this case.

Optimal Control for same IC with Robin BC

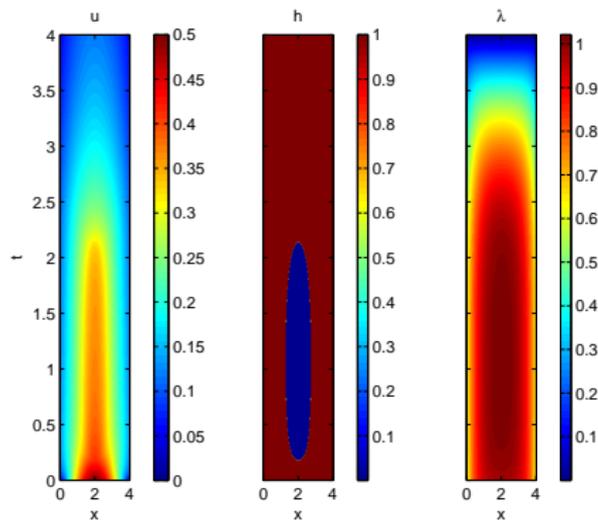


Figure: Final time 4, length of domain 4, discount .2, MORE RESERVE

These models are quite simple and only begin to investigate these issues.

Mike Neubert and Holly Moeller are investigating this when including habitat damage.

Choosing Movement Direction



Motivation: Second Part - Movement

- How resource allocation affects the population dynamics of species remains an important issue in conservation biology.
- Given a fixed amount of resources, how can we determine the optimal spatial arrangement of the favorable and unfavorable parts of the habitat for species to survive?
- This question was first addressed by Cantrell and Cosner

$$u_t = \lambda \Delta u + m(x)u - u^2 \quad \text{in } \Omega,$$

subject to Dirichlet, Robin, or Neumann BC

$u(x, t)$ is the density of the species

- $m(x)$ represents the intrinsic growth rate of the species and measures the availability of the resources.

A related question?

How does resource allocation affects population size of the species?

Population abundance is clearly a good measurement of conservation effort.

Control problem about Population Size

Given $0 < \delta < |\Omega|$, define the control set

$$U = \{m \in L^\infty(\Omega) \mid 0 \leq m(x) \leq 1, \int_{\Omega} m(x) \, dx = \delta\}.$$

We seek to find $m^* \in U$, such that

$$J(m^*) = \max_m J(m),$$

with objective functional

$$J(m) = \int_{\Omega} [u - (Bm^2)] \, dx, \quad (6)$$

$$\begin{cases} -\lambda \Delta u = mu - u^2, & x \in \Omega, \\ \frac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \quad (7)$$

paper with Ding, Finotti, Y. Lou, Y. Ye, in Nonlinear Analysis: Real World Applications 2010.

Numerical Illustrations

Used an iterative scheme to solve state and adjoint system with control characterization

Take $measure(\Omega) = 1$ and $\delta = .5$

1-D case

For $\lambda = .1$, $B > 1$ implies optimal control is constant.

Next we show $B = .5$ cases

See non-uniqueness and lack of symmetry

1-D case

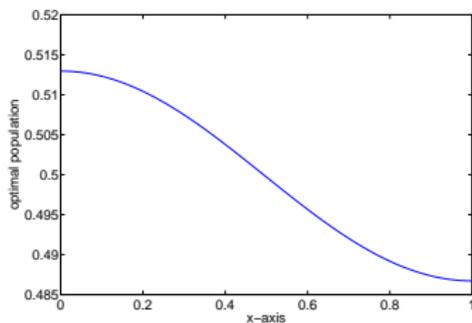
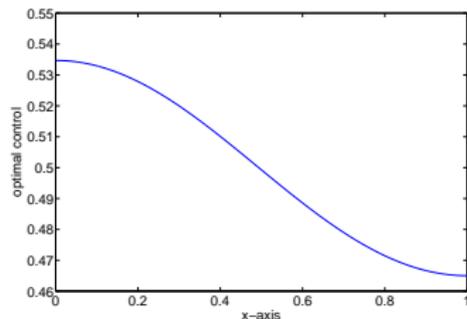


Figure: An OC and Corresponding State in 1D for $\lambda = 0.1$, $B = 0.5$

Another solution to same case

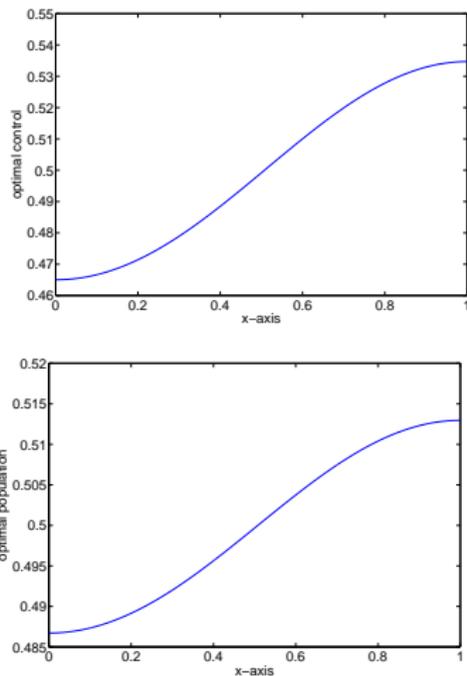


Figure: Another Optimal Control and Corresponding State in 1D for $\lambda = 0.1$, $B = 0.5$

2-D, optimal control concentrated at boundary

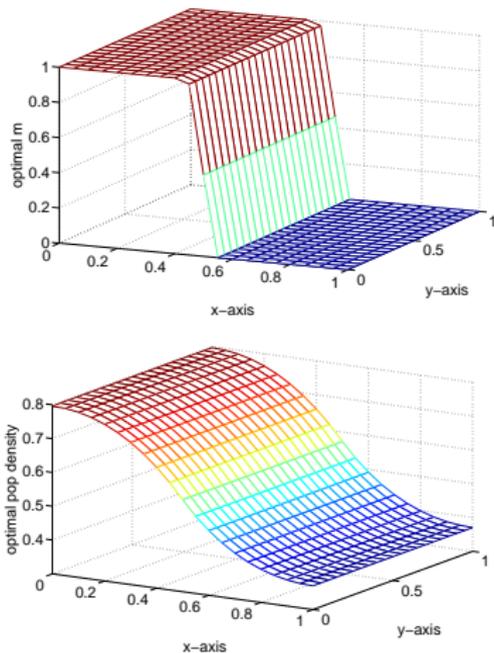


Figure: An OC and State in 2D for $\lambda = 0.1$, $B = 0.1$, $\delta = 0.5$

Conclusions from Some Numerical Results

- (i) For 1-D habitat, the characterization of OC depends on the choice of the diffusion rate λ . For small λ the OC seems to be symmetric, and so may be unique. This is in contrast to the case when λ is suitably larger, where OC is not unique and non-symmetric.

- (ii) For rectangular domains, the shape of OC depends on the choice of the amount of total resources, δ . When the amount is small, the OC is concentrated at one of the corners of the rectangle.

This is different from the situation where the amount of total resources is suitably large, for which the OC concentrates at a boundary edge of the rectangle.

Further investigation on relationships of λ , B and δ and on the parabolic case

Concentrating on Movement

- **Ecological question:** Given a fixed amount of resources, how does the species react to the habitat to be “beneficial”?
- Movement: Random Diffusion and Directed Advection.
- Belgacem-Cosner and Cantrell-Cosner-Lou studied the effects of the advection along an environmental resource gradient

$$u_t - \nabla \cdot [D\nabla u - \alpha u \nabla m(x)] = u[m(x) - u], \quad \Omega \times (0, \infty)$$

with zero flux boundary condition.

- $m(x)$ represents the intrinsic growth rate and measures the availability of the resources.
- “beneficial” means the persistence of the population or the existence of a unique globally attracting steady state.

A Related Question

If a species could choose the direction for advection movement, how would such a choice be made to maximize its total population?

Would the advection be related to the spatial gradient of m , the resource function? or the spatial gradient of $\ln(m)$?

Population Dynamics Model

- $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $Q_T = \Omega \times [0, T]$ and $S_T = \partial\Omega \times [0, T]$ for some fixed $T > 0$.
- Model with $u(x, t)$, population density

$$\begin{cases} u_t - \nabla \cdot [\mu \nabla u - u \vec{h}] & = & u[m - f(x, t, u)], & Q_T, \\ \mu \frac{\partial u}{\partial \nu} - u \vec{h} \cdot \nu & = & 0, & S_T, \\ u(\cdot, 0) & = & u_0 \geq 0, & \Omega. \end{cases}$$

- $\vec{h}: Q_T \rightarrow \mathbb{R}^n$ is the advection direction.
- $m = m(x, t)$ in $L^\infty(Q_T)$ measures the availability of resources.
- $f: Q_T \times \mathbb{R} \rightarrow \mathbb{R}$ is non-negative and satisfies some natural smoothness and growth conditions.
- $\mu > 0$ is fixed (diffusion coefficient).
- $u_0 \in L^\infty(\Omega)$ is sufficiently smooth.

Problem Formulation

- Seek the advection term $\vec{h}(x, t)$ **control** that maximizes the total population while minimizing the “cost” due to movement.
- Find $\vec{h}^* \in U$ such that

$$J(\vec{h}^*) = \max_U J(\vec{h}), \quad \text{where} \quad J(\vec{h}) = \int_{Q_T} [u(x, t) - B|\vec{h}(x, t)|^2] dx dt.$$

-

$$U = \{\vec{h} \in L^2((0, T), L^2(\Omega)^n) : |h_k| \leq M, \quad \forall k = 1, 2, \dots, n\}.$$

- B “cost coefficient” due to the population moving along \vec{h} .
- Denote the dependence of the state on the control by $u = u(\vec{h})$.

Existence Solutions and Some Estimates

Theorem

Given $m \in L^\infty(Q_T)$ and u_0 be non-negative, bounded and in $H^1(\Omega)$. Then, for each $\vec{h} \in U$, there is a unique weak solution $u = u(\vec{h})$ of

$$\begin{cases} u_t - \nabla \cdot [\mu \nabla u - u \vec{h}] & = u[m - f(x, t, u)], & Q_T, \\ \mu \frac{\partial u}{\partial \nu} - u \vec{h} \cdot \nu & = 0, & S_T, \\ u(\cdot, 0) & = u_0 \geq 0, & \Omega. \end{cases}$$

Moreover, there is a finite constant $C > 0$ such that

$$0 \leq u(\vec{h}) \leq C, \quad \forall (x, t) \in Q_T,$$

and

$$\sup_{0 \leq t \leq T} \int_{\Omega} u(x, t)^2 dx + \int_{Q_T} |\nabla u(x, t)|^2 dx dt \leq C.$$

Steps in the proof

- Solutions $u \geq 0$ follows from Stampacchia's truncation method (the standard maximum principle is not applicable here).
- The energy estimate

$$\sup_{0 \leq t \leq T} \int_{\Omega} u(x, t)^2 dx + \int_{Q_T} |\nabla u(x, t)|^2 dx dt \leq C.$$

follows by multiplying the equation with u and using Hölder's inequality, Sobolev embeddings.

- The upper bound for u , i.e. $u \leq C$ is not trivial. It follows from de Giorgi's iteration technique.
- The existence of solution follows by standard method (Galerkin's method).

Theorem

There exists an optimal control $\vec{h}^ \in U$ such that*

$$J(\vec{h}^*) = \max_{\vec{h} \in U} \int_{Q_T} [u(x, t, \vec{h}) - B|\vec{h}(x, t)|^2] dx dt.$$

- Careful analysis of the convergence of maximizing sequence of controls and corresponding states.
- The a-priori estimates of the solutions $u(\vec{h})$ are essential.

Characterize the Optimal Control

- To characterize the optimal solution \vec{h}^* , we need to differentiate the two maps
- the control-to-state map and then the control-to-objective functional map

$$\vec{h} \in U \rightarrow u(\vec{h}), \quad \text{and} \quad \vec{h} \in U \rightarrow J(\vec{h}).$$

Lemma

The mapping $\vec{h} \in U \rightarrow u(\vec{h})$ is differentiable in the following sense: for each \vec{h}, \vec{l} in U such that $\vec{h} + \epsilon \vec{l} \in U$ for all ϵ sufficiently small, there exists $\psi = \psi(\vec{h}, \vec{l}) \in L^2((0, T), H^1(\Omega))$, such that

$$\frac{u(\vec{h} + \epsilon \vec{l}) - u(\vec{h})}{\epsilon} \rightharpoonup \psi \text{ weakly in } L^2((0, T), H^1(\Omega)) \text{ as } \epsilon \rightarrow 0,$$

and the sensitivity ψ satisfies

$$\begin{cases} \psi_t - \nabla \cdot (\mu \nabla \psi - \vec{h} \psi) - [m - g(x, t, u)] \psi & = -\nabla \cdot (u \vec{l}), & Q_T, \\ \mu \frac{\partial \psi}{\partial \nu} - \psi \vec{h} \cdot \nu & = u \vec{l} \cdot \nu, & S_T, \\ \psi(x, 0) & = 0, & \Omega. \end{cases}$$

Here, $g(x, t, u) = \frac{\partial}{\partial u}[uf(x, t, u)]$.

Theorem

Given an optimal control \vec{h}^* and corresponding state $u^* = u(\vec{h}^*)$, there exists an adjoint solution p in $L^2(0, T, H^1(\Omega))$ which satisfies $p_t \in L^2((0, T), H^1(\Omega)^*)$ and

$$\begin{cases} -p_t - \mu \Delta p - \vec{h}^* \cdot \nabla p - [m - g(x, t, u^*)]p = 1, & \text{in } Q_T, \\ \frac{\partial p}{\partial \nu} = 0, & \text{in } S_T, \\ p(\cdot, T) = 0 & \text{in } \Omega. \end{cases}$$

Furthermore, \vec{h}^* is characterized by

$$h_i^* = \max \left\{ \min \left\{ M, \frac{u^* p_{x_i}}{2B} \right\}, -M \right\}, \quad \text{for each } i \in \{1, \dots, n\}.$$

- The operator in the p PDE is the adjoint of that in the ψ PDE (which is linear).

- The characterization of \vec{h}^* follows from the fact that

$$\lim_{\epsilon \rightarrow 0^+} \frac{J(\vec{h}^* + \epsilon \vec{l}) - J(\vec{h}^*)}{\epsilon} \leq 0$$

and by making use of the equations of adjoint and sensitivity equations.

- The L^∞ -bound of the solution $u(\vec{h})$ plays the key role in the analysis.

We are also interested in the questions

- The uniqueness of the optimal solutions \vec{h}^*
- The stability of the optimal solutions \vec{h}^* with respect to the given resource $m(x, t)$.

We now write

$$\vec{h}^* = \vec{h}^*(m).$$

Uniqueness and Stability Result

Theorem

Let $\beta > 0$. There exist $0 < T_1$ and B_1 such that if $B > B_1$ and $0 < T < T_1$, there exists a constant $C = C_T > 0$ such that the estimate

$$\|\vec{h}^*(m_1) - \vec{h}^*(m_2)\|_{L^2(Q_T)} \leq C \|m_1 - m_2\|_{L^2(Q_T)},$$

holds for all m_1, m_2 in $L^\infty(Q_T)$ with $|m_1|, |m_2| \leq \beta$.

Numerical Illustrations

- We use a forward-backward iterative scheme with finite difference method to solve the state and adjoint equations with optimal control characterization.
- We have run several examples for different types of nonlinearity f such as

$$f(x, t, u) = u, \quad f(x, t, u) = u + \frac{1}{1 + u}, \quad f(x, t, u) = u + \frac{u}{1 + u^2}.$$

- For each type of nonlinearity, we have run for both time-independent and time-dependent m .
- In all examples shown here,

$$\mu = 0.1, \quad , T = 0.2, \quad B = 0.05, \quad \text{and} \quad f(x, t, u) = u.$$

First Example

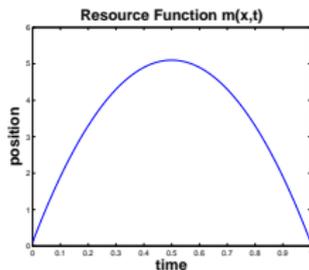


Figure: Time-Independent $m(x) = 20x(1 - x) + .1$

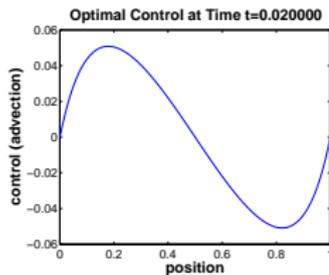
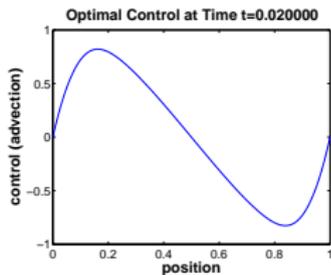


Figure: Time Slices of an Optimal Controls in 1D for $B = .05$ and $B = 1$

Second Example

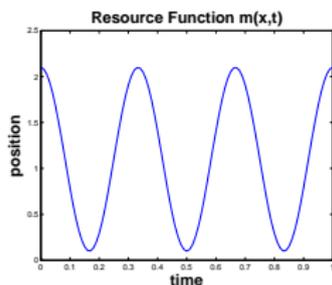


Figure: Time-Independent $m(x) = \cos(6\pi x) + 1.1$

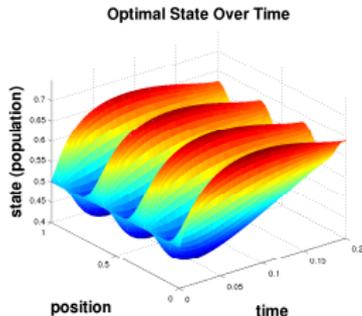
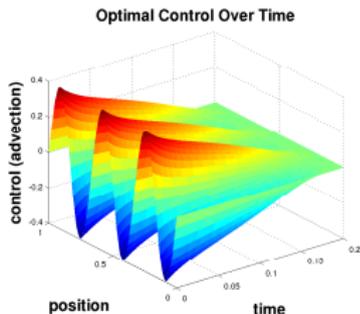


Figure: An Optimal Control and Corresponding State in 1D Over Time

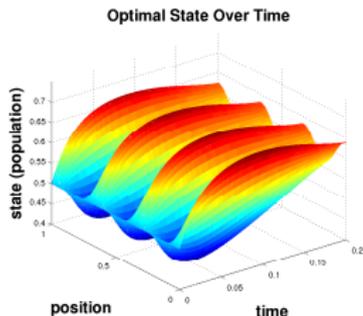
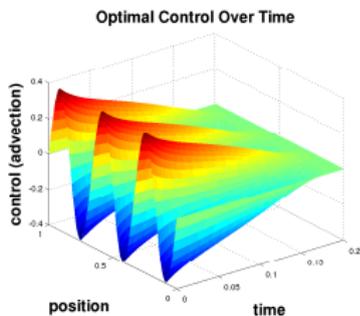


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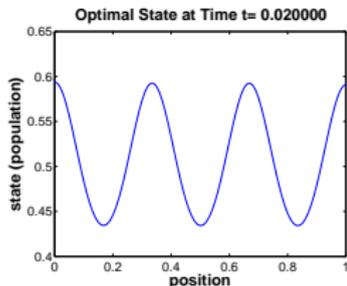
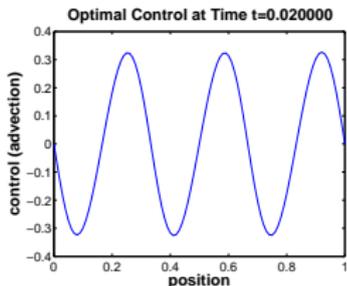


Figure: Time Slices of an Optimal Control and Corresponding State in 1D

Last Example

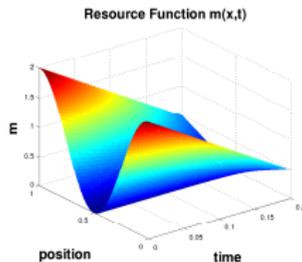


Figure: $m(x, t) = (1 - t/T)(\cos(2\pi x) + 1) + (t/T)|x - .5|$

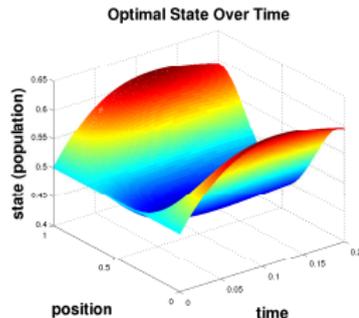
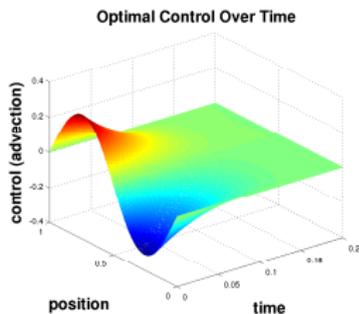


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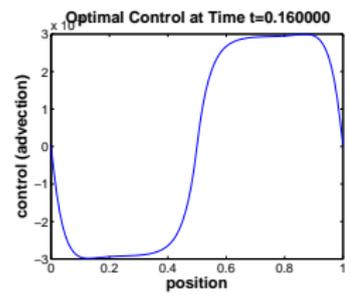
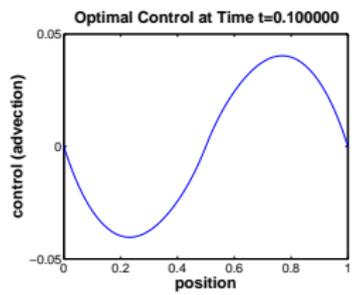
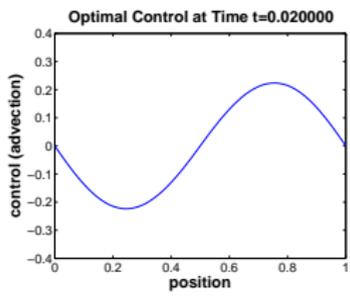


Figure: Early, Mid, and Late Time Slices of an Optimal Control and Corresponding m slice in 1D

Conclusions and Discussion

- We have proved the existence and characterized the optimal control.
- The uniqueness and stability of the optimal control are also obtained under some conditions on T and B .
- The numerical results indicate that the population follows the gradient of the given resource.
- Current work on elliptic case and ALSO investigate other relationships with the gradient of m

H. R. Joshi, G. E. Herrera, S. Lenhart, and M. Neubert, Optimal Dynamic Harvest of a Mobile Renewable Resource, *Natural Resource Modeling Journal* 22, 2009, 322-342.

H. Finotti, S. Lenhart, and T. Phan, Optimal control of advection director for reaction-diffusion population models, *Evolution Equations and Control Theory* 1 (2012), 81-107.

R. S. Cantrell and C. Cosner, Spatial Ecology, edited volume
Chapman and Hall, CRC Press, 2009.

R.S. Cantrell and C. Cosner, Spatial Ecology via Reaction-Diffusion
Equations, Series in Mathematical and Computational Biology,
John Wiley and Sons, Chichester, UK, 2003.

S. Lenhart and J. T. Workman, Optimal control applied to
Biological Models, Chapman and Hall/CRC, Boca Raton, 2007.

X. Li, X. and J. Yong, Optimal Control Theory for Infinite
Dimensional Systems, Birkhauser, 1995.



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