Competition, Diffusion, and Spatial Heterogeneity

Wei-Ming Ni

East China Normal University and University of Minnesota

December 15, 2012

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where $d > 0$, $u = u(x,t)$ and Ω is a bounded smooth domain in $\mathbb{R}^N$;

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Fact: *The unique steady state (s.s.) $u \equiv a$ is globally asymptotically stable (g.a.s.).*

Heterogeneous Environment

In a heterogeneous environment $m(x) \geq 0$, **nonconstant**

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- Observe that [Lou, 2006]

$$0 = d \int_\Omega \frac{|\nabla \theta_d|^2}{\theta_d^2} + \int_\Omega m - \int_\Omega \theta_d$$

$$\Rightarrow \int_\Omega \theta_d > \int_\Omega m(x) \qquad \forall d > 0, \text{ since } \theta_d \not\equiv const.$$

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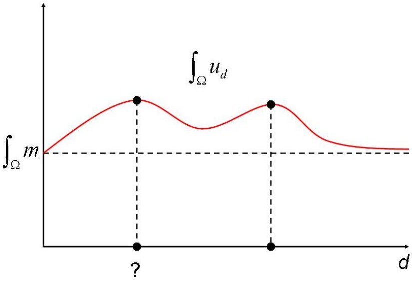

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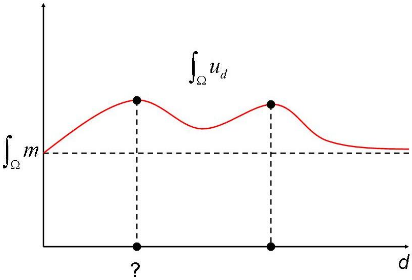

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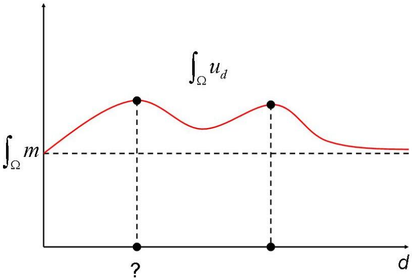

- What is the value $\max_{d>0} \int_\Omega \theta_d$?
- Where is $\max_{d>0} \int_\Omega \theta_d$ assumed?

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  (These questions have profound impacts on competition systems.)

Lotka-Volterra Competition

Lotka-Volterra competition system (ODE):

$$\begin{cases} U_t = U(a_1 - b_1 U - c_1 V) & \text{in } (0, T), \\ V_t = V(a_2 - b_2 U - c_2 V) & \text{in } (0, T). \end{cases}$$

- a_i: carrying capacity / intrinsic growth rate;
- b_1, c_2: intra-specific competition;
- b_2, c_1: inter-specific competition

  are all positive constants.

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- "Slower diffuser always prevails!"
- Here $m(x) \not\equiv$ *constant* is crucial!
- *Open:* If more than 2 competing species involved, it is **not known** if the slowest diffuser would prevail.

Weak Competition in Heterogeneous Environment

Consider special case $m_1 \equiv m_2$:

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Theorem (Lou; JDE (2006))

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Open: What happens when c is not small?

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The rest of this talk is based on my joint work with Xiaoqing He.

Principal Eigenvalue

$$\begin{cases} \Delta\varphi + \lambda h(x)\varphi = 0 & \text{in } \Omega, \\ \partial_\nu \varphi = 0 & \text{on } \partial\Omega, \end{cases}$$

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Lemma

The problem has a nonzero principal eigenvalue $\lambda_1 = \lambda_1(h)$ iff h changes sign and $\int_\Omega h \neq 0$. More precisely, if h changes sign, then

1. $\int_\Omega h = 0 \Leftrightarrow$ 0 *is the only principal eigenvalue.*
2. $\int_\Omega h > 0 \Leftrightarrow$ $\lambda_1(h) < 0$.
3. $\int_\Omega h < 0 \Leftrightarrow$ $\lambda_1(h) > 0$.
4. $\lambda_1(h_1) > \lambda_1(h_2)$ *if* $h_1 \leq h_2$, $h_1 \not\equiv h_2$, *and* h_1, h_2 *both change sign.*
5. $\lambda_1(h)$ *is continuous in* h*; i.e.* $\lambda_1(h_\ell) \to \lambda_1(h)$ *if* $h_\ell \to h$ *in* $L^\infty(\Omega)$.

Spatial Concentration/Variation

Let ρ_m be the unique solution of

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- (LC): $C(m) > 0$

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- For step function $m = \tau\chi_{[0,(1/\tau)]}$, we have $C(m) > 0 \Leftrightarrow \tau > 3$

Q2: Homogeneity vs Heterogeneity

(Q2): *More fundamentally, the effect of spatial heterogeneity:* What if $m_1 \not\equiv$ const but $m_2(x) \equiv$ const and $\int_\Omega m_1 = \int_\Omega m_2$?

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Consider, for simplicity, denote m_1 by m, and $m_2 \equiv \overline{m} = 1$,

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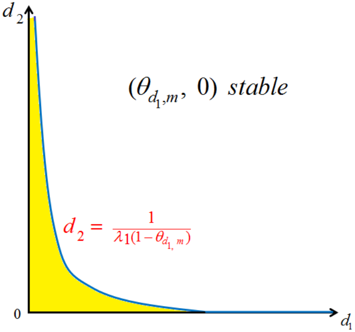

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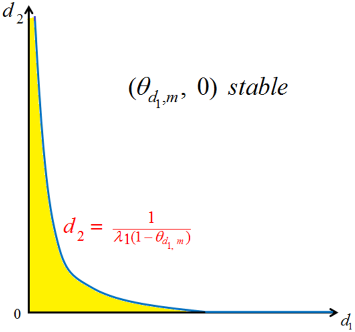

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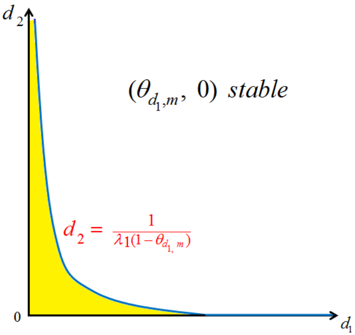

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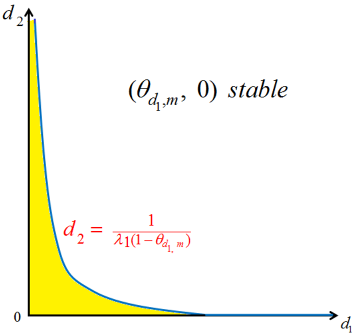

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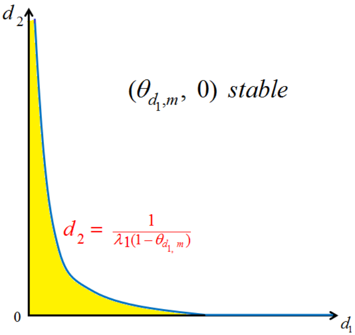

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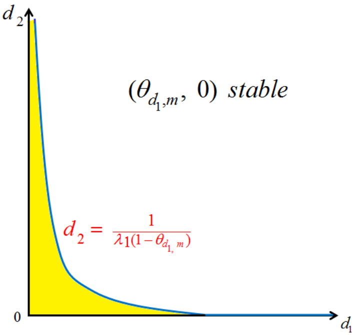

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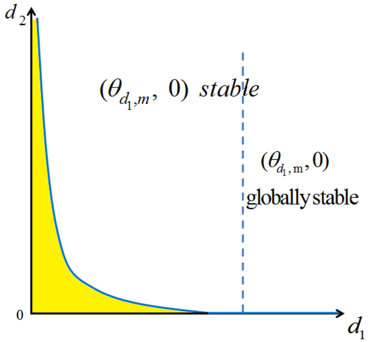

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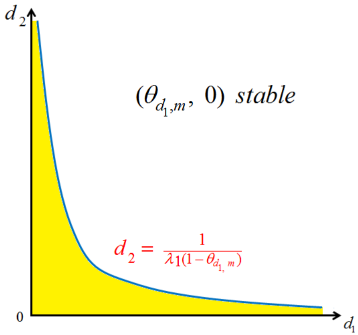

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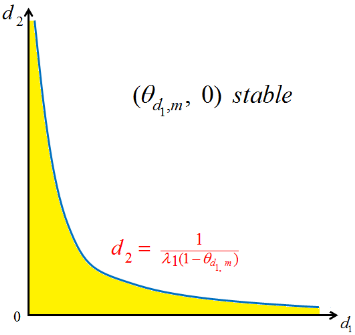

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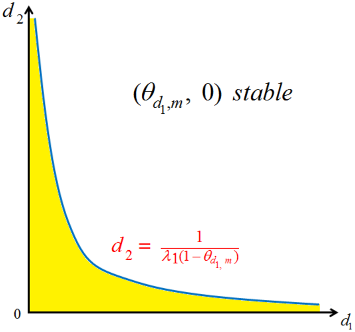

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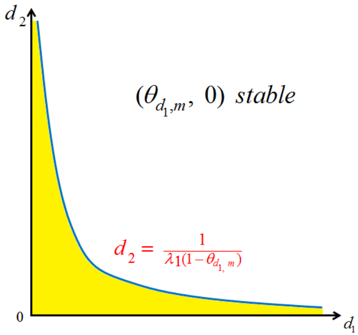

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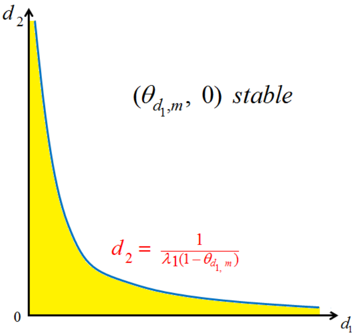

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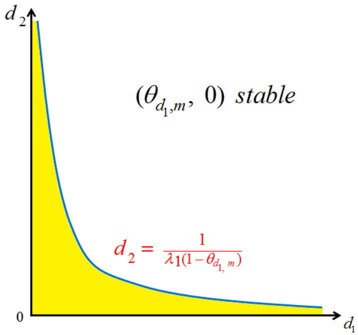

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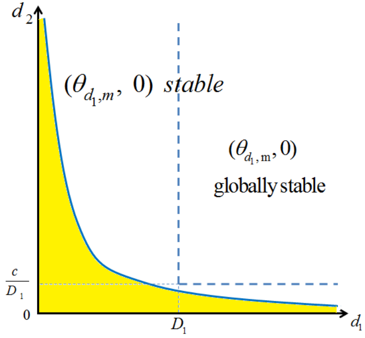

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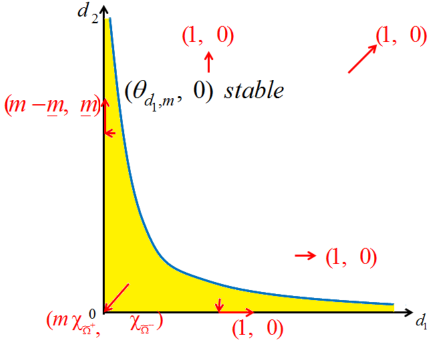

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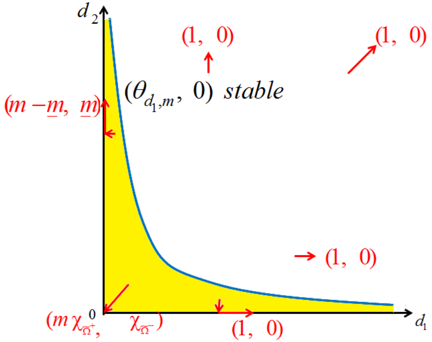

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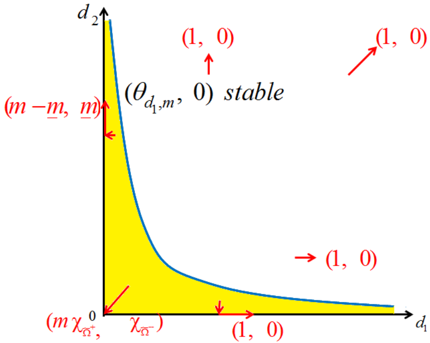

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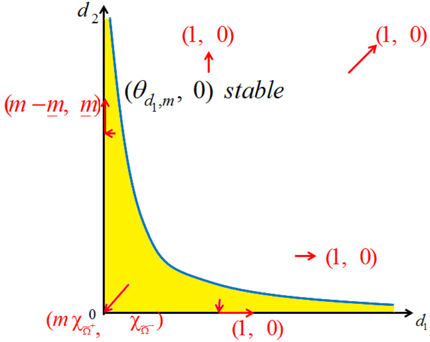

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- $\underline{m} = \inf_\Omega m$.

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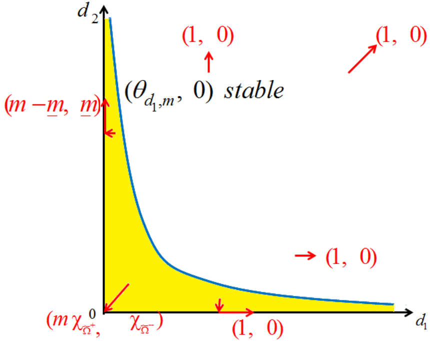
$$\begin{cases} U_t = d_1 \Delta U + U(m(x) - U - V) & \text{in } \Omega \times \mathbb{R}^+, \\ V_t = d_2 \Delta V + V(1 - U - V) & \text{in } \Omega \times \mathbb{R}^+, \\ \partial_\nu U = \partial_\nu V = 0 & \text{on } \partial\Omega \times \mathbb{R}^+, \\ U(x,0) = U_0(x), \quad V(x,0) = V_0(x) & \text{in } \Omega, \end{cases}$$



- Yellow region denotes stable co-existence s.s.
- Red arrows are directions of limits for s.s.
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- $\tilde{\Omega}^{+(-)} = \{m > (<) 1\}$ (assuming $|\{m = 1\}| = 0$).

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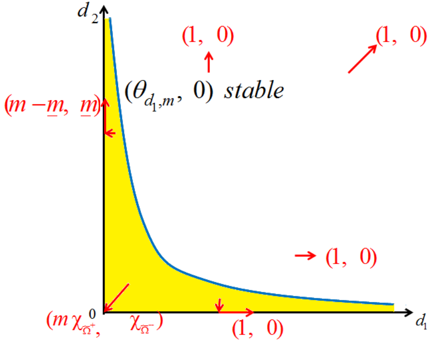

- Yellow region denotes stable co-existence s.s.
- Red arrows are directions of limits for s.s.
- $\underline{m} = \inf_{\Omega} m$.
- $\tilde{\Omega}^{+(-)} = \{m > (<) 1\}$ (assuming $|\{m = 1\}| = 0$).
- For d_1 large, U tends to dominate, regardless.

Q2: $m \neq 1 = \overline{m}$

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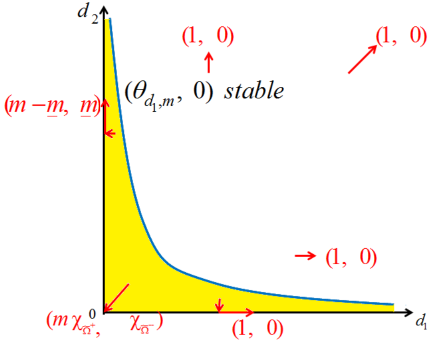
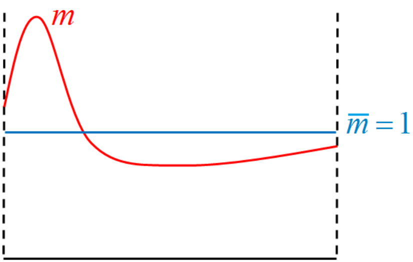


Figure : $(m, 1)$



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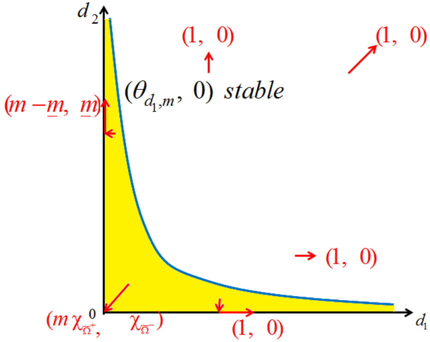
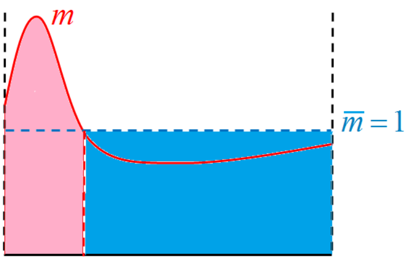


Figure : $(m\chi_{\tilde{\Omega}^+}, \chi_{\tilde{\Omega}^-})$



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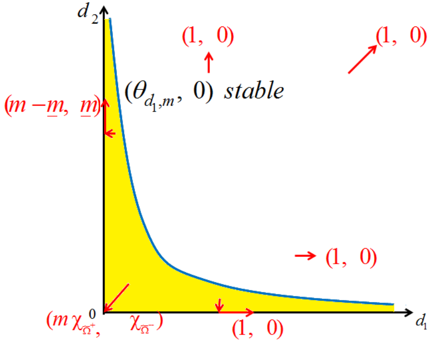

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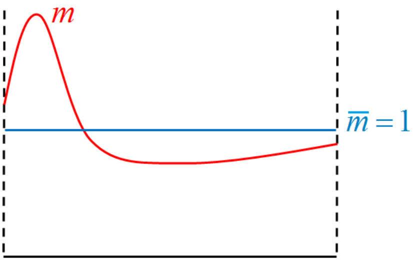

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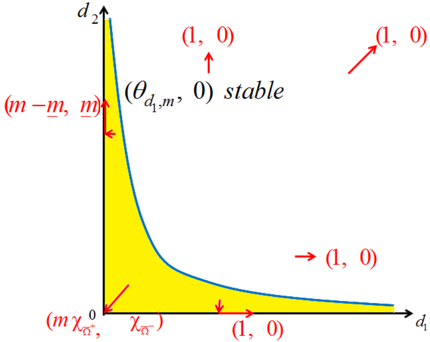
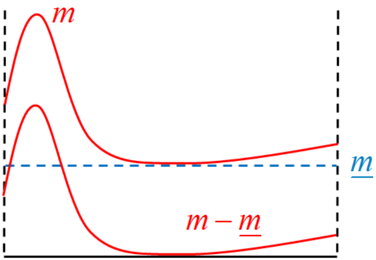


Figure : $(m - \underline{m}, \underline{m})$



Q2: Homogeneity vs Heterogeneity $m \not\equiv 1 = \overline{m}$

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Recap:

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- Heterogeneous $m(x)$ seems ”superior” to homogeneous ones.

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- Heterogeneous $m(x)$ seems "superior" to homogeneous ones.
- For U with heterogeneous $m(x)$, the larger the concentration (of $m(x)$), the better!

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Recap:

- Heterogeneous $m(x)$ seems "superior" to homogeneous ones.
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- For U with heterogeneous $m(x)$, the larger the diffusion rate d_1, the better!

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  (This seems *interesting, especially compared to "Slower diffuser always prevails!"*)

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- For U with heterogeneous $m(x)$, the larger the diffusion rate d_1, the better!
  (This seems *interesting, especially compared to "Slower diffuser always prevails!"*)
- On the other hand, for any d_1 fixed, $(\theta_{d_1,m}, 0)$ is globally asymp stable for all d_2 large; i.e. *U still prevails if d_2 is large.*

Proof of $\lim_{d_2 \to \infty} \lim_{d_1 \to 0^+} (U_{d_1,d_2}, V_{d_1,d_2}) = (m - \underline{m}, \underline{m})$

Step I: $\lim_{d_1 \to 0^+} (U_{d_1,d_2}, V_{d_1,d_2}) := (U_{0,d_2}, V_{0,d_2})$, where

$$\|V_{0,d_2}\|_{L^\infty(\Omega)} > \inf_{\Omega} m,$$

for any $d_2 > 0$.

Proof of $\lim_{d_2\to\infty}\lim_{d_1\to 0^+}(U_{d_1,d_2}, V_{d_1,d_2}) = (m-\underline{m}, \underline{m})$

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for any $d_2 > 0$. For, otherwise, as $d_1 \to 0$

$$\text{equation for } U \Rightarrow U_{0,d_2} = m - V_{0,d_2},$$

$$\text{equation for } V \Rightarrow 1 - U_{d_1,d_2} - V_{d_1,d_2} \to (1-m) \text{ uniformly on } \bar{\Omega}.$$

Proof of $\lim_{d_2\to\infty}\lim_{d_1\to 0^+}(U_{d_1,d_2}, V_{d_1,d_2}) = (m-\underline{m}, \underline{m})$

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However, again by equation for V, as $d_1 \to 0$,

$$d_2^{-1} = \lambda_1(1 - U_{d_1,d_2} - V_{d_1,d_2}) \to \lambda_1(1-m) = 0,$$

a contradiction (since d_2 is arbitrary but fixed and finite so far).

Proof of $\lim_{d_2\to\infty}\lim_{d_1\to 0^+}(U_{d_1,d_2}, V_{d_1,d_2}) = (m - \underline{m}, \underline{m})$

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Step II: $\lim_{d_2\to\infty}\lim_{d_1\to 0^+}(U_{d_1,d_2}, V_{d_1,d_2}) = (m - \xi, \xi)$, for some

$$0 \le \xi \le \inf_\Omega m.$$

Proof of $\lim_{d_2\to\infty}\lim_{d_1\to 0^+}(U_{d_1,d_2},V_{d_1,d_2}) = (m-\underline{m},\underline{m})$

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Since $\lim_{d_2\to\infty}\lim_{d_1\to 0^+}(U_{d_1,d_2},V_{d_1,d_2}) = (U_{0,\infty},V_{0,\infty})$, where $V_{0,\infty}\equiv\xi\ge 0$,

Proof of $\lim\limits_{d_2\to\infty}\lim\limits_{d_1\to 0^+}(U_{d_1,d_2}, V_{d_1,d_2}) = (m - \underline{m}, \underline{m})$

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$$U_{0,\infty} = (m-\xi)^+ \text{ and } \int_\Omega (1-(m-\xi)^+ - \xi) = 0.$$

Proof of $\lim_{d_2\to\infty}\lim_{d_1\to 0^+}(U_{d_1,d_2},V_{d_1,d_2}) = (m-\underline{m},\underline{m})$

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Thus $\quad \overline{m} = 1 \Rightarrow \xi \le \inf_\Omega m.$

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Since $\lim\limits_{d_2\to\infty}\lim\limits_{d_1\to 0^+}(U_{d_1,d_2}, V_{d_1,d_2}) = (U_{0,\infty}, V_{0,\infty})$, where $V_{0,\infty} \equiv \xi \ge 0$, we have

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Thus $\quad \overline{m} = 1 \Rightarrow \xi \le \inf\limits_\Omega m.$

Since Step I $\Rightarrow \xi \ge \inf\limits_\Omega m, \qquad \lim\limits_{d_2\to\infty}\lim\limits_{d_1\to 0^+} V_{d_1,d_2} = \inf\limits_\Omega m \equiv \underline{m}.$

Q1: $m_1 \neq m_2$ & $\int_\Omega m_1 = \int_\Omega m_2$

$$
\begin{cases}
U_t = d_1 \Delta U + U(m_1(x) - U - V) & \text{in } \Omega \times \mathbb{R}^+, \\
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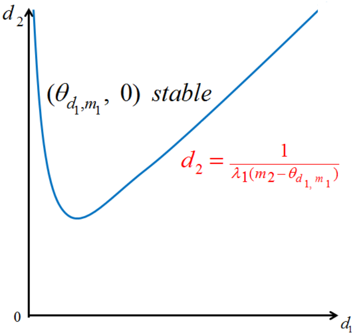

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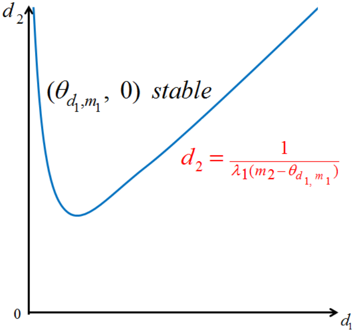

- The blue curve separates (locally) linearly stable and unstable regions for $(\theta_{d_1,m_1}, 0)$.

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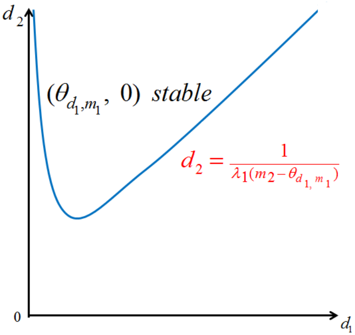

- The blue curve separates (locally) linearly stable and unstable regions for $(\theta_{d_1,m_1}, 0)$.
- *Limits of blue curve*: Both are ∞ when $d_1 \to 0$ or ∞.

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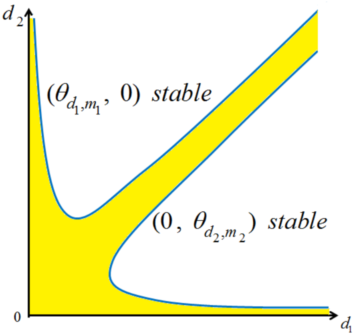

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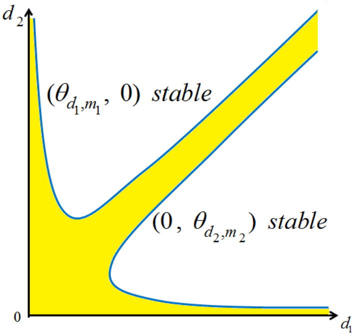

- The two white regions are disjoint and never touch each other!

Q1: $m_1 \not\equiv m_2$ & $\int_\Omega m_1 = \int_\Omega m_2$

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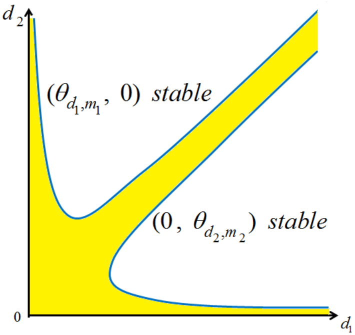

- The two white regions are disjoint and never touch each other!
- Yellow region denotes stable co-existence s.s.

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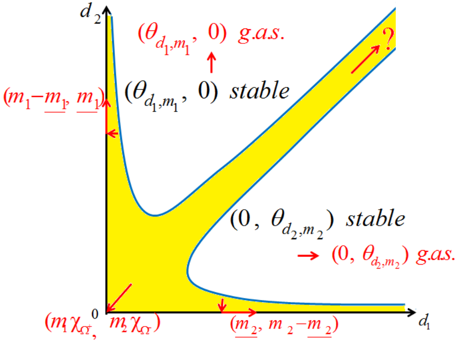

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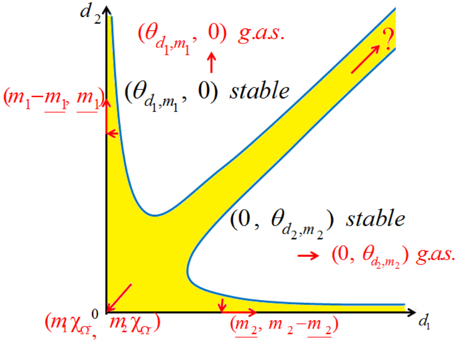

- Red arrows denote directions of limits for s.s.

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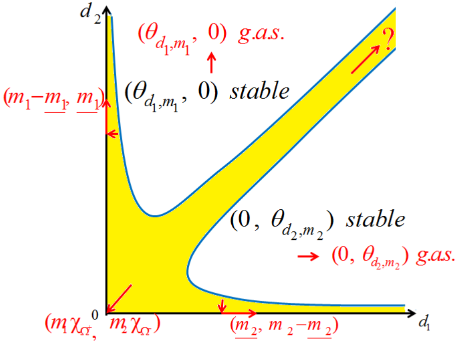

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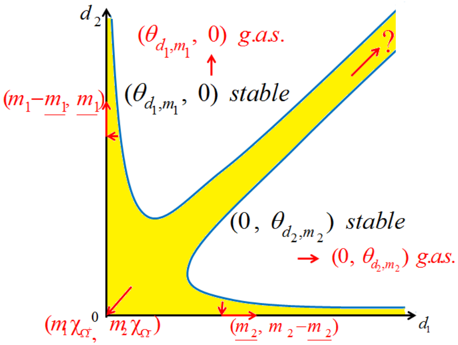

- Red arrows denote directions of limits for s.s.
- $\underline{m_i} = \inf_\Omega m_i$, $i = 1, 2$.
- $\Omega^{+(-)} = \{m_1 > (<) m_2\}$ (assuming $|\{m_1 = m_2\}| = 0$).

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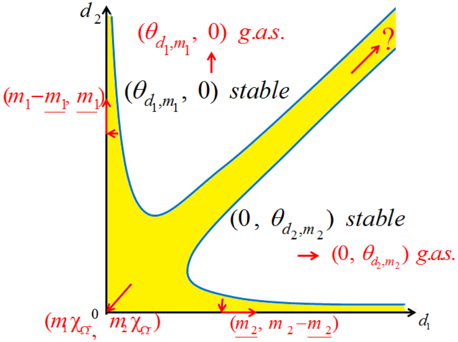
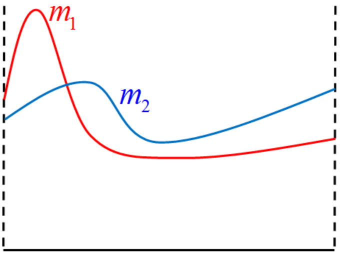


Figure : $(m_1 \chi_{\Omega^+}, m_2 \chi_{\Omega^-})$



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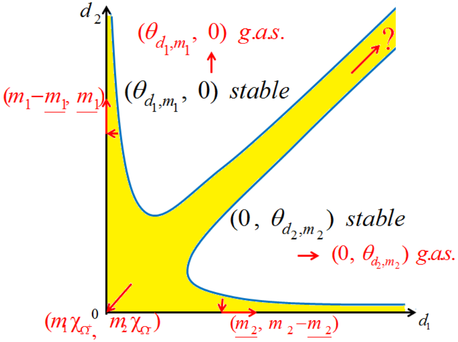

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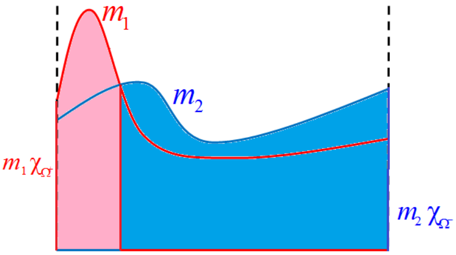

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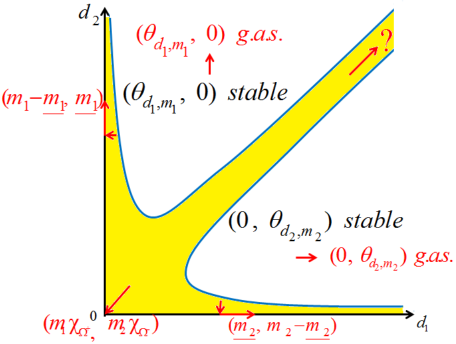
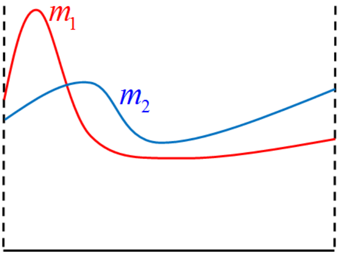


Figure : $(\underline{m_2}, m_2 - \underline{m_2})$



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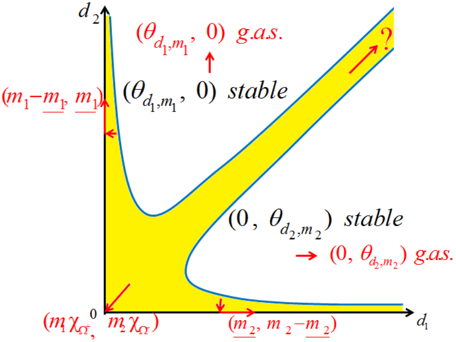

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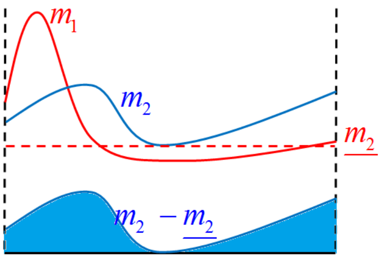

Recap:

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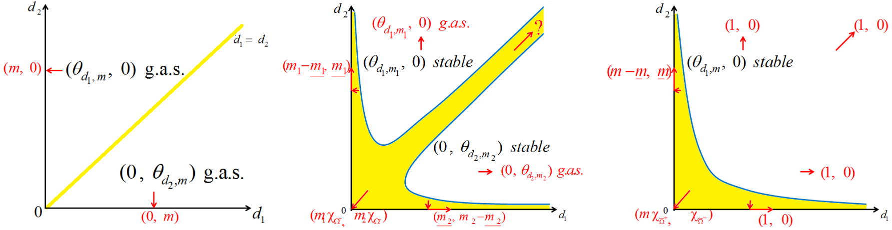

Stability of $(\theta_{d_1,m_1}, 0)$

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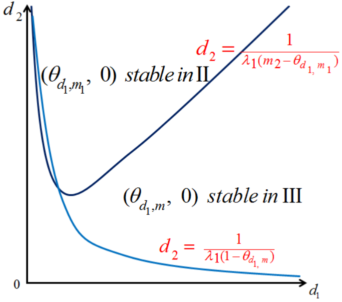

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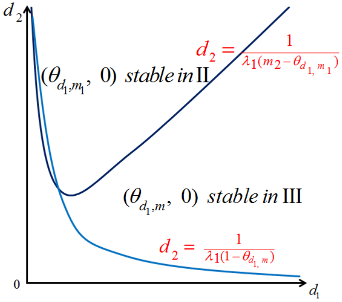

$\lim_{d_1 \to \infty} \lambda_1(m_2 - \theta_{d_1,m_1}) = 0$

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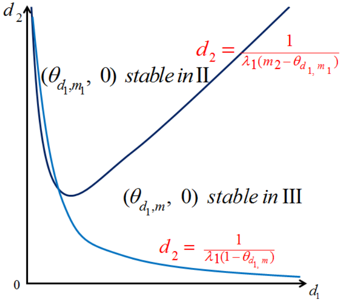

$\lim_{d_1 \to \infty} \lambda_1(m_2 - \theta_{d_1,m_1}) = 0$, $\lim_{d_1 \to \infty} \lambda_1(1 - \theta_{d_1,m}) = \infty$. Why?

Linearized eigenvalue problem

We first observe the 1st eigenvalue of

$$\begin{cases} d\Delta\psi + h(x)\psi + \mu\psi = 0 & \text{in } \Omega \\ \partial_\nu \psi = 0 & \text{on } \partial\Omega \end{cases}$$

is given by

$$\mu_1(d,h) = \inf_{\psi \in H^1(\Omega)} \left\{ \frac{\int_\Omega [d|\nabla\psi|^2 - h(x)\psi^2]}{\int_\Omega \psi^2} \right\}$$

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(a) $\mu_1(d,h)$ *is strictly* $\uparrow$ *in* d *and,* $h(x) \gneqq k(x) \Rightarrow \mu_1(d,h) < \mu_1(d,k)$.

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Note: $\mu_1(d,h) = 0 \Leftrightarrow d = 1/\lambda_1(h)$

Sketch of Proofs

$$\begin{cases} U_t = d_1 \Delta U + U(m_1(x) - U - V) & \text{in } \Omega \times \mathbb{R}^+, \\ V_t = d_2 \Delta V + V(m_2(x) - U - V) & \text{in } \Omega \times \mathbb{R}^+, \\ \partial_\nu U = \partial_\nu V = 0 & \text{on } \partial\Omega \times \mathbb{R}^+. \end{cases}$$

Linearize the system at $(\theta_{d_1,m_1}, 0)$:

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$$\begin{cases} d_1 \Delta \Psi_1 + \Psi_1(m_1 - 2\theta_{d_1,m_1}) - \theta_{d_1,m_1}\Psi_2 + \lambda \Psi_1 = 0 & \text{in } \Omega \\ d_2 \Delta \Psi_2 + \Psi_2(m_2 - \theta_{d_1,m_1}) + \lambda \Psi_2 = 0 & \text{in } \Omega \\ \partial_\nu \Psi_1 = \partial_\nu \Psi_2 = 0 & \text{on } \partial\Omega. \end{cases} \tag{L1}$$

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Claim: Stability of (L1) = stability of (S1).
Thus led to the region $d_2 > 1/\lambda_1(m_2 - \theta_{d_1,m_1})$ in Case II,
while in Case III$(m_1 = m, m_2 \equiv 1)$, $d_2 > 1/\lambda_1(1 - \theta_{d_1,m})$.

Sketch of Proofs

$\lim_{d_1\to\infty} \lambda_1(m_2 - \theta_{d_1,m_1}) = 0, \quad \lim_{d_1\to\infty} \lambda_1(1 - \theta_{d_1,m}) = \infty.$

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Recall property of $\lambda_1(\cdot)$: $\int_\Omega h = 0 \Leftrightarrow$ 0 is the only principal eigenvalue; $\lambda_1(h)$ is continuous in h.

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$$\Delta\varphi + \lambda_1 \cdot (1 - \theta)\varphi = 0 \text{ in } \Omega, \quad \partial_\nu \varphi = 0 \text{ on } \partial\Omega,$$

$$d_1 \Delta\theta + (m - \theta)\theta = 0 \text{ in } \Omega, \quad \partial_\nu \theta = 0 \text{ on } \partial\Omega.$$

$$\begin{aligned} 0 &= \int_\Omega |\nabla\varphi|^2 + \lambda_1 \int_\Omega (\theta - 1)\varphi^2 \\ &= \int_\Omega |\nabla\varphi|^2 + \lambda_1 \int_\Omega (\theta - \bar\theta)(\varphi - \bar\varphi)(\varphi + \bar\varphi) + \lambda_1 \int_\Omega (\bar\theta - \bar m)\varphi^2 \end{aligned}$$

Sketch of Proofs

$\lim_{d_1\to\infty}\lambda_1(m_2-\theta_{d_1,m_1})=0, \quad \lim_{d_1\to\infty}\lambda_1(1-\theta_{d_1,m})=\infty.$

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$$\begin{aligned}0&=\int_\Omega|\nabla\varphi|^2+\lambda_1\int_\Omega(\theta-1)\varphi^2\\&=\int_\Omega|\nabla\varphi|^2+\lambda_1\int_\Omega(\theta-\bar\theta)(\varphi-\bar\varphi)(\varphi+\bar\varphi)+\lambda_1\int_\Omega(\bar\theta-\bar m)\varphi^2\\&>\int_\Omega|\nabla\varphi|^2-\frac{\lambda_1\|\varphi+\bar\varphi\|_\infty}{2}\int_\Omega[(\theta-\bar\theta)^2+(\varphi-\bar\varphi)^2]+\frac{\lambda_1 d_1\overline{\varphi^2}}{\|m\|_\infty^2}\int_\Omega|\nabla\theta|^2\end{aligned}$$

Sketch of Proofs

$\lim_{d_1 \to \infty} \lambda_1(m_2 - \theta_{d_1,m_1}) = 0, \quad \lim_{d_1 \to \infty} \lambda_1(1 - \theta_{d_1,m}) = \infty.$

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$$\begin{aligned}
0 &= \int_\Omega |\nabla\varphi|^2 + \lambda_1 \int_\Omega (\theta - 1)\varphi^2 \\
&= \int_\Omega |\nabla\varphi|^2 + \lambda_1 \int_\Omega (\theta - \bar\theta)(\varphi - \bar\varphi)(\varphi + \bar\varphi) + \lambda_1 \int_\Omega (\bar\theta - \bar m)\varphi^2 \\
&> \int_\Omega |\nabla\varphi|^2 - \frac{\lambda_1 \|\varphi + \bar\varphi\|_\infty}{2} \int_\Omega [(\theta - \bar\theta)^2 + (\varphi - \bar\varphi)^2] + \frac{\lambda_1 d_1 \overline{\varphi^2}}{\|m\|_\infty^2} \int_\Omega |\nabla\theta|^2 \\
&> \left(1 - \frac{\lambda_1 C \|\varphi + \bar\varphi\|_\infty}{2}\right) \int_\Omega |\nabla\varphi|^2 + \lambda_1 \left(\frac{d_1 \overline{\varphi^2}}{\|m\|_\infty^2} - \frac{C \|\varphi + \bar\varphi\|_\infty}{2}\right) \int_\Omega |\nabla\theta|^2
\end{aligned}$$

Sketch of Proofs

$\lim_{d_1\to\infty}\lambda_1(m_2-\theta_{d_1,m_1})=0, \quad \lim_{d_1\to\infty}\lambda_1(1-\theta_{d_1,m})=\infty.$

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&> \left(1-\frac{\lambda_1 C\|\varphi+\bar\varphi\|_\infty}{2}\right)\int_\Omega|\nabla\varphi|^2+\lambda_1\left(\frac{d_1\overline{\varphi^2}}{\|m\|_\infty^2}-\frac{C\|\varphi+\bar\varphi\|_\infty}{2}\right)\int_\Omega|\nabla\theta|^2\\
&> 0,
\end{aligned}$$

Sketch of Proofs

$\lim_{d_1\to\infty} \lambda_1(m_2 - \theta_{d_1,m_1}) = 0, \quad \lim_{d_1\to\infty} \lambda_1(1 - \theta_{d_1,m}) = \infty.$

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&> \left(1 - \frac{\lambda_1 C \|\varphi + \bar\varphi\|_\infty}{2}\right) \int_\Omega |\nabla\varphi|^2 + \lambda_1 \left(\frac{d_1 \overline{\varphi^2}}{\|m\|_\infty^2} - \frac{C\|\varphi+\bar\varphi\|_\infty}{2}\right) \int_\Omega |\nabla\theta|^2 \\
&> 0,
\end{aligned}$$

if $\lambda_1 \to \lambda^* \in [0,\infty)$, since $\varphi, \bar\varphi \to$ const (chosen suitably small) uniformly on $\bar\Omega$.

Sketch of Proofs

$\lim_{d_1\to\infty}\lambda_1(m_2-\theta_{d_1,m_1})=0, \quad \lim_{d_1\to\infty}\lambda_1(1-\theta_{d_1,m})=\infty.$

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&> \left(1-\frac{\lambda_1 C\|\varphi+\bar\varphi\|_\infty}{2}\right)\int_\Omega|\nabla\varphi|^2+\lambda_1\left(\frac{d_1\overline{\varphi^2}}{\|m\|_\infty^2}-\frac{C\|\varphi+\bar\varphi\|_\infty}{2}\right)\int_\Omega|\nabla\theta|^2 \\
&> 0,
\end{aligned}$$

if $\lambda_1\to\lambda^*\in[0,\infty)$, since $\varphi,\bar\varphi\to$ const (chosen suitably small) uniformly on $\bar\Omega$. Contradiction! Thus $\lim_{d_1\to\infty}\lambda_1(1-\theta_{d_1,m})=\infty$. □