# Dynamical Properties of cosmological solutions

George Ellis

January 1, 2004

**Abstract**

I will consider domains of dependence in theory and in practice, showing how while there are nice FOSH formulations and associated results in relativistic cosmology, the charasteristics of scalar and vector perturbations are timelike, hence the real domains of dependence in cosmology are much smaller than implied by the light cone. This can be clearly seen in relation to the different physical conditions at different stages of evolution of the universe. Finally I will give some brief comments about the nature of physically important gravitational effects at these different stages and put the question: are there any epochs in the evolution of realistic universe models where tensor (gravitational wave) modes are important? This will be distinguished from the Bianchi Type IX oscillations which are 'silent universes' characterised by ODE's. The issue here is what characterises a gravitational wave; I suggest that the criterion is that both $curlE \neq 0$ and $curlH \neq 0$.In appendices I will briefly relate this to the issue of the arrow of time and how that problem looks different at each of these epochs.

## 1 Domains of dependence: theory and practice

The purpose now is to look at domains of dependence in cosmological models and their relation to irreversible local processes and the arrow of time. We shall see that the theoretical domains of dependence are different than the practical ones. . Finally we ask if there are ever times in the history of the universe when gravitational wave effects are significant, as opposed to time-dependent tidal forces. At the present time gravitational wave effects are insignificant: and that is why a local Newtonian approximation, and indeed the existence of isolated systems, is possible. In Appendices, we briefly look at how this relates to irreversible local processes and the arrow of time.

## 2 First-order symmetric hyperbolic systems

We consider evolution systems for a collection of $k$ real-valued field variables $u^A = u^A(x^\mu)$ that are composed of a set of $k$ quasi-linear partial differential equations of *first order* given by

$$M^{AB\,\mu}(x^\nu, u^C)\,\partial_\mu u_B = N^A(x^\nu, u^C)\ , \quad A,\, B,\, C = 1, \ldots, k\ ; \tag{1}$$

the field variables $u^A$ are functions of a set of local spacetime coordinates $\{ x^\mu \}$. Evolution systems of this form are called *symmetric* if the real-valued $k \times k$ coefficient matrices entering the principle part satisfy $M^{AB\,\mu} = M^{(AB)\,\mu}$; moreover, they are called *hyperbolic* if the contraction $M^{AB\,\mu}\, n_\mu$ with the coordinate components of an arbitrary *past-directed* timelike 1-form $n_a$ yields a positive-definite matrix. Thus these are FOSH (first-order symmetric hyperbolic) systems [7], [3]. We remark that (i) cases with $M^{AB\,\mu} = M^{AB\,\mu}(x^\nu)$ are referred to as *semi-linear*, and (ii) cases with $M^{AB\,\mu} = M^{AB\,\mu}(u^C)$ and $N^A = N^A(u^C)$ are referred to as *autonomous*. In general, it proves convenient to consider a (1+3)–decomposition of Eq. (1) in the format

$$M^{AB\,0}(t, x^j, u^C)\, \partial_t u_B + M^{AB\,i}(t, x^j, u^C)\, \partial_i u_B = N^A(t, x^j, u^C) \ .$$

The *characteristic condition*

$$0 = Q := \det [\, M^{AB\,\mu}\, \nabla_\mu \phi \,] \tag{2}$$

determines the coordinate components of the *past-directed* normals $\nabla_a \phi$ of the set of *characteristic 3-surfaces* $\mathcal{C}$:$\{\phi(x^\mu)$ =const$\}$ associated with the FOSH evolution system (1). With $M^{AB\,\mu} = M^{(AB)\,\mu}$, hyperbolicity of Eq. (1) thus also corresponds to all individual roots ("eigenvalues") $v$ of Eq. (2) being *real-valued*. Every individual $v$ then defines a pair of *left* and *right nullifying vectors*, $l^A$ and $r^A$, by

$$0 =: l_A\, (M^{AB\,\mu}\, \nabla_\mu \phi) \ , \quad 0 =: (M^{AB\,\mu}\, \nabla_\mu \phi)\, r_B \ ; \tag{3}$$

the linearly independent sets $\{ l^A \}$ or $\{ r^A \}$ form a basis of the $k$-dimensional space of field variables $u^A$.

According to the theory discussed in Ch. VI.4.2 of Courant and Hilbert [3], FOSH evolution systems of the format (1) have the power to describe the physical transport along *bicharacteristic rays* of *jump discontinuities* that exist in the *outward first derivatives* across a characteristic 3-surface $\mathcal{C}$:$\{\phi(x^\mu)$ =const$\}$ of the field variables $u^A$; the tangential first derivatives of the $u^A$ as well as the $u^A$ themselves are assumed to be *continuous* across $\mathcal{C}$:$\{\phi(x^\mu)$ =const$\}$. As is standard, we will use the notation

$$[f] := \lim_{\phi \to c_+} f - \lim_{\phi \to c_-} f = f_+ - f_-$$

to symbolise a jump discontinuity across $\mathcal{C}$:$\{\phi(x^\mu)$ =const$\}$ in the value of a given variable $f$. Under the stated assumptions, it follows from Eqs. (1) and (3) that

$$0 = (M^{AB\,\mu}\, \nabla_\mu \phi)\, [\, \partial_\phi u_B \,] \quad \Leftrightarrow \quad [\, \partial_\phi u^A \,] = [\partial_\phi u]\, r^A \ , \tag{4}$$

i.e., the jump discontinuity $[\, \partial_\phi u^A \,]$ must be proportional to a right nullifying vector $r^A$. The real-valued scalar of proportionality, denoted by $[\partial_\phi u]$, is assumed to have continuous first derivatives. Then, according to Chs. VI.4.2 and VI.4.9 of Ref. [3], for linear, semi-linear and quasi-linear FOSH evolution systems (1) the *transport equation* for $[\partial_\phi u]$ along bicharacteristic rays within the characteristic 3-surfaces $\mathcal{C}$:$\{\phi(x^\mu) = \text{const}\}$ takes the effective form

$$0 = (l_A M^{AB\,\mu} r_B)\,\partial_\mu[\partial_\phi u] + \big((l_A M^{AB\,\mu})\,\partial_\mu r_B - (l_A N^{AB} r_B)\big)\,[\partial_\phi u]\ . \qquad (5)$$

Note, in particular, the *involutive character* of this relation; if $[\partial_\phi u]$ is non-zero at one point along a bicharacteristic ray, it will be non-zero everywhere along this ray. Note also that the present treatment of jump discontinuities breaks down when the $\mathcal{C}$:$\{\phi(x^\mu) = \text{const}\}$ within a given family start to intersect and so prompt the formation of shocks. Shock formation, however, cannot arise when the principal part of Eq. (1) is semi-linear. It is a special feature of the relativistic gravitational field equations that related FOSH evolution systems have principal parts which are effectively semi-linear, in the branches that evolve the degrees of freedom in the gravitational field itself.

We now put the cosmological equations discussed in the last lecture into FOSH form, following [1, 2].

## 2.1 Choice of gauge source functions and coordinates

There exists within a $1+3$ orthonormal frame representation of the relativistic gravitational field equations a set of ten *gauge source functions*, $G := \{\,T^0,\, T^\alpha,\, T^\alpha{}_0,\, T^\alpha{}_\beta\,\}$, that can be *arbitrarily prescribed* in any dynamical consideration (and are thus assumed to be "known") [6]. These relate to (i) the arbitrary choice of a *future-directed* reference "time flow vector field" $\mathbf{T}$, which, in terms of the orthonormal basis $\{\,\mathbf{e}_0,\, \mathbf{e}_\alpha\,\}$, is expressed by

$$\mathbf{T} := T^0\,\mathbf{e}_0 + T^\alpha\,\mathbf{e}_\alpha\ , \quad T^0 > 0\ , \qquad (6)$$

and (ii) the propagation of the orthonormal basis $\{\,\mathbf{e}_0,\, \mathbf{e}_\alpha\,\}$ along $\mathbf{T}$, described by

$$\nabla_{\mathbf{T}}\mathbf{e}_0 := T^\alpha{}_0\,\mathbf{e}_\alpha\ , \quad \nabla_{\mathbf{T}}\mathbf{e}_\alpha := T^0{}_\alpha\,\mathbf{e}_0 + T^\beta{}_\alpha\,\mathbf{e}_\beta\ . \qquad (7)$$

Parallel transport of $\{\,\mathbf{e}_0,\, \mathbf{e}_\alpha\,\}$ along $\mathbf{T}$ corresponds to setting $0 = T^0{}_\alpha = T^\alpha{}_\beta$. Upon introduction of a dimensionless local time coordinate $t$ and dimensionless local spatial coordinates $\{\,x^i\,\}$ that comove with $\mathbf{T}$, the *gauge conditions* related to a $1+3$ orthonormal frame representation are made explicit by [6]

$$e_0{}^\mu = \frac{1}{T^0}\,(M_0^{-1}\,\delta^\mu{}_0 - T^\alpha\,e_\alpha{}^\mu)\ , \quad \Gamma^0{}_{\alpha 0} = \frac{1}{T^0}\,(T^0{}_\alpha - \Gamma^0{}_{\alpha\beta}\,T^\beta)\ , \qquad (8)$$

$$\Gamma^\alpha{}_{\beta 0} = \frac{1}{T^0}\,(T^\alpha{}_\beta - \Gamma^\alpha{}_{\beta\gamma}\,T^\gamma)\ ; \qquad (9)$$

we keep the inverse unit of [ *length* ], $M_0^{-1}$, as a coefficient for reasons of physical dimensions.

The fluid-comoving, *Lagrangean* perspective where we identify the timelike reference congruence with the fluid 4-velocity field, $\mathbf{e}_0 \equiv \mathbf{u}$, is now obtained by fixing three of the four dimensionless coordinate gauge source functions according to $T^{\alpha} = 0$, resulting in an alignment $\mathbf{T} \parallel \mathbf{e}_0$ ($\equiv \mathbf{u}$). This leads to

$$e_0{}^{\mu} = M^{-1}\,\delta^{\mu}{}_0 \ , \quad T^0{}_{\alpha} = T^0\,\Gamma^0{}_{\alpha 0} = T^0\,\dot{u}_{\alpha} \ , \quad T^{\alpha}{}_{\beta} = T^0\,\Gamma^{\alpha}{}_{\beta 0} = T^0\,\epsilon^{\alpha}{}_{\beta\gamma}\,\Omega^{\gamma} \ , \tag{10}$$

where $M := T^0 M_0$. Consequently, the three frame gauge source functions $T^0{}_{\alpha}$ become proportional to the components of the fluid acceleration $\dot{u}^{\alpha}$, while the three frame gauge source functions $T^{\alpha}{}_{\beta}$ become proportional to the components of the rotation rate $\Omega^{\alpha}$ at which the spatial frame $\{\,\mathbf{e}_{\alpha}\,\}$ fails to be Fermi-propagated along $\mathbf{u}$.

## 2.2 Connection components and commutators

The 24 algebraically independent frame components of the spacetime connection $\Gamma^{a}{}_{bc}$ can be split into the set

$$\Gamma_{\alpha 00} = \dot{u}_{\alpha} = \Gamma_{F\,00\alpha} \tag{11}$$

$$\Gamma_{\alpha 0\beta} = \frac{1}{3}\,\Theta\,\delta_{\alpha\beta} + \sigma_{\alpha\beta} - \epsilon_{\alpha\beta\gamma}\,\omega^{\gamma} = \Gamma_{F\,0\beta\alpha} \tag{12}$$

$$\Gamma_{\alpha\beta 0} = \epsilon_{\alpha\beta\gamma}\,\Omega^{\gamma} = \Gamma_{F\,0\alpha\beta} \tag{13}$$

$$\Gamma_{\alpha\beta\gamma} = 2\,a_{[\alpha}\,\delta_{\beta]\gamma} + \epsilon_{\gamma\delta[\alpha}\,n^{\delta}{}_{\beta]} + \frac{1}{2}\,\epsilon_{\alpha\beta\delta}\,n^{\delta}{}_{\gamma} = \Gamma_{F\,\beta\gamma\alpha} \ . \tag{14}$$

It contains the familiar kinematical fluid variables: $\dot{u}^{\alpha}$, its relativistic acceleration, $\Theta$, its rate of expansion, $\sigma_{\alpha\beta} = \sigma_{\beta\alpha}$, its rate of shear (with $\sigma^{\alpha}{}_{\alpha} = 0$), and $\omega^{\alpha}$, its vorticity. Moreover, $\Omega^{\alpha}$ is the rate of rotation of the spatial frame $\{\,\mathbf{e}_{\alpha}\,\}$ with respect to a Fermi-propagated basis. Finally, $a^{\alpha}$ and $n_{\alpha\beta} = n_{\beta\alpha}$ are 9 spatial commutation functions.

The condition that the spacetime connection be torsion-free, $\nabla_{[a}\nabla_{b]}f = 0$ for any spacetime scalar $f$, then translates into the commutator equations

$$[\,\mathbf{e}_0, \mathbf{e}_{\alpha}\,](f) = \dot{u}_{\alpha}\;\mathbf{e}_0(f) - [\;\frac{1}{3}\,\Theta\,\delta_{\alpha}{}^{\beta} + \sigma_{\alpha}{}^{\beta} + \epsilon_{\alpha}{}^{\beta}{}_{\gamma}\,(\omega^{\gamma} + \Omega^{\gamma})\;]\;\mathbf{e}_{\beta}(f) \tag{15}$$

$$[\,\mathbf{e}_{\alpha}, \mathbf{e}_{\beta}\,](f) = 2\,\epsilon_{\alpha\beta\gamma}\,\omega^{\gamma}\;\mathbf{e}_0(f) + [\;2\,a_{[\alpha}\,\delta^{\gamma}{}_{\beta]} + \epsilon_{\alpha\beta\delta}\,n^{\delta\gamma}\;]\;\mathbf{e}_{\gamma}(f) \ . \tag{16}$$

## 2.3 Constraint equations

The following relations in the set obtained from an extended $1+3$ orthonormal frame representation of the relativistic gravitational field equations do *not* contain any frame derivatives with respect to $\mathbf{e}_0$. Hence, we refer to these relations as "constraint equations". They are [5]

$$\begin{aligned} 0 = (C_1)^{\alpha} &:= (\mathbf{e}_{\beta} - 3\,a_{\beta})\,(\sigma^{\alpha\beta}) - \frac{2}{3}\,\delta^{\alpha\beta}\,\mathbf{e}_{\beta}(\Theta) \\ &\quad - n^{\alpha}{}_{\beta}\,\omega^{\beta} + \epsilon^{\alpha\beta\gamma}\,[\;(\mathbf{e}_{\beta} + 2\,\dot{u}_{\beta} - a_{\beta})\,(\omega_{\gamma}) - n_{\beta\delta}\,\sigma^{\delta}{}_{\gamma}\;] \end{aligned} \tag{17}$$

$$0 = (C_2) := (\mathbf{e}_\alpha - \dot{u}_\alpha - 2\,a_\alpha)\,(\omega^\alpha) \quad (18)$$

$$0 = (C_3)^{\alpha\beta} := (\delta^{\gamma\langle\alpha}\,\mathbf{e}_\gamma + 2\,\dot{u}^{\langle\alpha} + a^{\langle\alpha\rangle})\,(\omega^{\beta\rangle}) - \frac{1}{2}\,n^\gamma{}_\gamma\,\sigma^{\alpha\beta} + 3\,n^{\langle\alpha}{}_\gamma\,\sigma^{\beta\rangle\gamma} + H^{\alpha\beta} - \epsilon^{\gamma\delta\langle\alpha}\,[\,(\mathbf{e}_\gamma - a_\gamma)\,(\sigma^{\beta\rangle}{}_\delta) + n^{\beta\rangle}{}_\gamma\,\omega_\delta\,] \quad (19)$$

$$0 = (C_{\rm J})^\alpha := (\mathbf{e}_\beta - 2\,a_\beta)\,(n^{\alpha\beta}) + \frac{2}{3}\,\Theta\,\omega^\alpha + 2\,\sigma^\alpha{}_\beta\,\omega^\beta + \epsilon^{\alpha\beta\gamma}\,[\,\mathbf{e}_\beta(a_\gamma) - 2\,\omega_\beta\,\Omega_\gamma\,] \quad (20)$$

$$0 = (C_{\rm G})^{\alpha\beta} := {}^*S^{\alpha\beta} + \frac{1}{3}\,\Theta\,\sigma^{\alpha\beta} - \sigma^{\langle\alpha}{}_\gamma\,\sigma^{\beta\rangle\gamma} - \omega^{\langle\alpha}\,\omega^{\beta\rangle} + 2\,\omega^{\langle\alpha}\,\Omega^{\beta\rangle} - E^{\alpha\beta} \quad (21)$$

$$0 = (C_{\rm G}) := {}^*R + \frac{2}{3}\,\Theta^2 - (\sigma_{\alpha\beta}\sigma^{\alpha\beta}) + 2\,(\omega_\alpha\omega^\alpha) - 4\,(\omega_\alpha\,\Omega^\alpha) - 2\,\mu - 2\,\Lambda \quad (22)$$

$$0 = (C_4)^\alpha := (\mathbf{e}_\beta - 3\,a_\beta)\,(E^{\alpha\beta}) - \frac{1}{3}\,\delta^{\alpha\beta}\,\mathbf{e}_\beta(\mu) - 3\,\omega_\beta\,H^{\alpha\beta} - \epsilon^{\alpha\beta\gamma}\,[\,\sigma_{\beta\delta}\,H^\delta{}_\gamma + n_{\beta\delta}\,E^\delta{}_\gamma\,] \quad (23)$$

$$0 = (C_5)^\alpha := (\mathbf{e}_\beta - 3\,a_\beta)\,(H^{\alpha\beta}) + (\mu + p)\,\omega^\alpha + 3\,\omega_\beta\,E^{\alpha\beta} + \epsilon^{\alpha\beta\gamma}\,[\,\sigma_{\beta\delta}\,E^\delta{}_\gamma - n_{\beta\delta}\,H^\delta{}_\gamma\,] \quad (24)$$

$$0 = (C_{\rm PF})^\alpha := c_s^2\,\delta^{\alpha\beta}\,\mathbf{e}_\beta(\mu) + (\mu + p)\,\dot{u}^\alpha\ ,$$

where

$${}^*S_{\alpha\beta} := \mathbf{e}_{\langle\alpha}(a_{\beta\rangle}) + b_{\langle\alpha\beta\rangle} - \epsilon^{\gamma\delta}{}_{\langle\alpha}\,(\mathbf{e}_{|\gamma|} - 2\,a_{|\gamma|})\,(n_{\beta\rangle\delta}) \quad (25)$$

$${}^*R := 2\,(2\,\mathbf{e}_\alpha - 3\,a_\alpha)\,(a^\alpha) - \frac{1}{2}\,b^\alpha{}_\alpha \quad (26)$$

$$b_{\alpha\beta} := 2\,n_{\alpha\gamma}\,n_\beta{}^\gamma - n^\gamma{}_\gamma\,n_{\alpha\beta}\ , \quad (27)$$

$c_s^2(\mu) := dp(\mu)/d\mu$ defines the *isentropic speed of sound* with $0 \leq c_s^2 \leq 1$, and angle brackets denote the symmetric tracefree part. When $0 = \omega^\alpha(\mathbf{u})$, such that the fluid 4-velocity field $\mathbf{u}$ constitutes the normals to a family of spacelike 3-surfaces $\mathcal{S}$:$\{t =$const$\}$. In this case, one also speaks of $(C_G)$ as the generalised Friedmann equation, alias the "Hamiltonian constraint" or the "energy constraint".

## 2.4 Propagation equations

These are the equations with explicit time derivatives. They are [5]

#### 2.4.1 Evolution of spatial commutation functions

The 9 spatial commutation functions $a^\alpha$ and $n_{\alpha\beta}$ are generally evolved by Equations originating from the Jacobi identity. Employing each of the constraints $(C_1)^\alpha$ to $(C_3)^{\alpha\beta}$ listed in the previous paragraph, we can eliminate $\mathbf{e}_\alpha$ frame derivatives of the kinematical fluid variables $\Theta$, $\sigma_{\alpha\beta}$ and $\omega^\alpha$ from their right-hand sides. Thus, we obtain the following equations for the evolution of the spatial connection components:

$$\begin{aligned}\mathbf{e}_0(a^\alpha) &= -\frac{1}{3}(\Theta\,\delta^\alpha{}_\beta - \frac{3}{2}\,\sigma^\alpha{}_\beta)\,(\dot{u}^\beta + a^\beta) + \frac{1}{2}\,n^\alpha{}_\beta\,\omega^\beta - \frac{1}{2}\,q^\alpha \\ &\quad - \frac{1}{2}\,\epsilon^{\alpha\beta\gamma}\,[\,(\dot{u}_\beta + a_\beta) - n_{\beta\delta}\,\sigma^\delta{}_\gamma - (\mathbf{e}_\beta + \dot{u}_\beta - 2\,a_\beta)\,(\Omega_\gamma)\,] + \frac{1}{2}\,(C_1)^\alpha\end{aligned} \tag{28}$$

$$\begin{aligned}\mathbf{e}_0(n^{\alpha\beta}) &= -\frac{1}{3}\,\Theta\,n^{\alpha\beta} - \sigma^{\langle\alpha}{}_\gamma\,n^{\beta\rangle\gamma} + \frac{1}{2}\,\sigma^{\alpha\beta}\,n^\gamma{}_\gamma - (\dot{u}^{\langle\alpha} + a^{\langle\alpha})\,\omega^{\beta\rangle} - H^{\alpha\beta} + (\delta^{\gamma\langle\alpha}\,\mathbf{e}_\gamma + \dot{u}^{\langle\alpha})\,(\Omega^{\beta\rangle}) \\ &\quad - \frac{2}{3}\,\delta^{\alpha\beta}\,[\,2\,(\dot{u}_\gamma + a_\gamma)\,\omega^\gamma - \sigma^\gamma{}_\delta\,n^\delta{}_\gamma + (\mathbf{e}_\gamma + \dot{u}_\gamma)\,(\Omega^\gamma)\,] \\ &\quad - \epsilon^{\gamma\delta\langle\alpha}\,[\,(\dot{u}_\gamma + a_\gamma)\,\sigma^{\beta\rangle}{}_\delta - (\omega_\gamma + 2\,\Omega_\gamma)\,n^{\beta\rangle}{}_\delta\,] - \frac{2}{3}\,\delta^{\alpha\beta}\,(C_2) + (C_3)^{\alpha\beta}\ .\end{aligned} \tag{29}$$

#### 2.4.2 Evolution of kinematical fluid variables

The evolution equations for the 9 kinematical fluid variables $\Theta$, $\sigma_{\alpha\beta}$ and $\omega^\alpha$ are provided by the familiar Ricci field equations,

$$\mathbf{e}_0(\Theta) - \mathbf{e}_\alpha(\dot{u}^\alpha) = -\frac{1}{3}\,\Theta^2 + (\dot{u}_\alpha - 2\,a_\alpha)\,\dot{u}^\alpha - (\sigma^\alpha{}_\beta\sigma^\beta{}_\alpha) + 2\,(\omega_\alpha\omega^\alpha) - \frac{1}{2}\,(\mu + 3p) + \Lambda \tag{30}$$

$$\begin{aligned}\mathbf{e}_0(\sigma^{\alpha\beta}) - \delta^{\gamma\langle\alpha}\,\mathbf{e}_\gamma(\dot{u}^{\beta\rangle}) &= -\frac{2}{3}\,\Theta\,\sigma^{\alpha\beta} - \sigma^{\langle\alpha}{}_\gamma\,\sigma^{\beta\rangle\gamma} - \omega^{\langle\alpha}\,\omega^{\beta\rangle} + (\dot{u}^{\langle\alpha} + a^{\langle\alpha})\,\dot{u}^{\beta\rangle} - (E^{\alpha\beta} - \frac{1}{2}\,\pi^{\alpha\beta}) \\ &\quad + \epsilon^{\gamma\delta\langle\alpha}\,[\,2\,\Omega_\gamma\,\sigma^{\beta\rangle}{}_\delta - n^{\beta\rangle}{}_\gamma\,\dot{u}_\delta\,]\end{aligned} \tag{31}$$

$$\mathbf{e}_0(\omega^\alpha) - \frac{1}{2}\,\epsilon^{\alpha\beta\gamma}\,\mathbf{e}_\beta(\dot{u}_\gamma) = -\frac{2}{3}\,\Theta\,\omega^\alpha + \sigma^\alpha{}_\beta\,\omega^\beta - \frac{1}{2}\,n^\alpha{}_\beta\,\dot{u}^\beta - \frac{1}{2}\,\epsilon^{\alpha\beta\gamma}\,[\,a_\beta\,\dot{u}_\gamma - 2\,\Omega_\beta\,\omega_\gamma\,]\ . \tag{32}$$

#### 2.4.3 Evolution of Weyl curvature and matter variables

Finally, we have the Bianchi field equations for the 10 Weyl curvature variables $E_{\alpha\beta}$ and $H_{\alpha\beta}$ and the 4 matter variables $\mu$ and $q^\alpha$, which are obtained from the once-contracted and twice-contracted second Bianchi identity, respectively:

$$\begin{aligned}\mathbf{e}_0(E^{\alpha\beta} + \frac{1}{2}\,\pi^{\alpha\beta}) - \epsilon^{\gamma\delta\langle\alpha}\,\mathbf{e}_\gamma(H^{\beta\rangle}{}_\delta) &+ \frac{1}{2}\,\delta^{\gamma\langle\alpha}\,\mathbf{e}_\gamma(q^{\beta\rangle}) = \\ &- \frac{1}{2}\,(\mu + p)\,\sigma^{\alpha\beta} - \Theta\,(E^{\alpha\beta} + \frac{1}{6}\,\pi^{\alpha\beta}) + 3\,\sigma^{\langle\alpha}{}_\gamma\,(E^{\beta\rangle\gamma} - \frac{1}{6}\,\pi^{\beta\rangle\gamma}) \\ &+ \frac{1}{2}\,n^\gamma{}_\gamma\,H^{\alpha\beta} - 3\,n^{\langle\alpha}{}_\gamma\,H^{\beta\rangle\gamma} - \frac{1}{2}\,(2\,\dot{u}^{\langle\alpha} + a^{\langle\alpha})\,q^{\beta\rangle}\end{aligned} \tag{33}$$

$$
\begin{aligned}
& + \epsilon^{\gamma\delta\langle\alpha} \, [ \, (2\,\dot{u}_\gamma - a_\gamma)\, H^{\beta\rangle}{}_\delta \\
& + (\omega_\gamma + 2\,\Omega_\gamma)\,(E^{\beta\rangle}{}_\delta + \frac{1}{2}\,\pi^{\beta\rangle}{}_\delta) + \frac{1}{2}\, n^{\beta\rangle}{}_\gamma\, q_\delta \, ]
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{e}_0(H^{\alpha\beta}) + \epsilon^{\gamma\delta\langle\alpha}\, \mathbf{e}_\gamma(E^{\beta\rangle}{}_\delta - \frac{1}{2}\,\pi^{\beta\rangle}{}_\delta) \;=\; & - \Theta\, H^{\alpha\beta} + 3\,\sigma^{\langle\alpha}{}_\gamma\, H^{\beta\rangle\gamma} + \frac{3}{2}\,\omega^{\langle\alpha}\, q^{\beta\rangle} \\
& - \frac{1}{2}\, n^\gamma{}_\gamma\,(E^{\alpha\beta} - \frac{1}{2}\,\pi^{\alpha\beta}) + 3\, n^{\langle\alpha}{}_\gamma\,(E^{\beta\rangle\gamma} - \frac{1}{2}\,\pi^{\beta\rangle\gamma}) \\
& + \epsilon^{\gamma\delta\langle\alpha}\, [ \, a_\gamma\,(E^{\beta\rangle}{}_\delta - \frac{1}{2}\,\pi^{\beta\rangle}{}_\delta) - 2\,\dot{u}_\gamma\, E^{\beta\rangle}{}_\delta \\
& + \frac{1}{2}\,\sigma^{\beta\rangle}{}_\gamma\, q_\delta + (\omega_\gamma + 2\,\Omega_\gamma)\, H^{\beta\rangle}{}_\delta \, ]
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
\mathbf{e}_0(q^\alpha) + \delta^{\alpha\beta}\,\mathbf{e}_\beta(p) + \mathbf{e}_\beta(\pi^{\alpha\beta}) \;=\; & -\frac{4}{3}\,\Theta\, q^\alpha - \sigma^\alpha{}_\beta\, q^\beta - (\mu + p)\,\dot{u}^\alpha - (\dot{u}_\beta - 3\,a_\beta)\,\pi^{\alpha\beta} \\
& - \epsilon^{\alpha\beta\gamma}\, [ \, (\omega_\beta - \Omega_\beta)\, q_\gamma - n_{\beta\delta}\,\pi^\delta{}_\gamma \, ]
\end{aligned} \tag{35}
$$

$$
\mathbf{e}_0(\mu) + \mathbf{e}_\alpha(q^\alpha) \;=\; - \Theta\,(\mu + p) - 2\,(\dot{u}_\alpha - a_\alpha)\, q^\alpha - \sigma^\alpha{}_\beta\,\pi^\beta{}_\alpha \; . \tag{36}
$$

### 2.5 Matter model

The matter source will be assumed to be a *perfect fluid* such that, with respect to fluid-comoving observers,

$$
0 = q^\alpha(\mathbf{u}) = \pi_{\alpha\beta}(\mathbf{u}) \; , \tag{37}
$$

i.e., the energy current density and the anisotropic pressure both vanish. Additionally, a *baryotropic* equation of state is assumed,

$$
p = p(\mu) \; , \tag{38}
$$

relating the isotropic pressure $p(\mathbf{u})$ to the total energy density $\mu(\mathbf{u})$. As above, $c_s^2 := dp(\mu)/d\mu$ is the isentropic *speed of sound*. Under the assumptions (37) the evolution equation for $q^\alpha$ reduces to the new constraint equation

$$
0 = (C_{PF})^\alpha := \delta^{\alpha\beta}\,\mathbf{e}_\beta(p) + (\mu + p)\,\dot{u}^\alpha \; , \tag{39}
$$

which is often called the momentum conservation equation.

The general extended $1+3$ ONF dynamical equations just given do not directly form a FOSH evolution system, for two reasons. First, they do *not* provide evolution equations for any of the geometrical variables $\dot{u}^\alpha$, $\Omega^\alpha$, $p$ and $\pi_{\alpha\beta}$.[1] In the case of $\Omega^\alpha$ this is a reflection of the freedom of choice of a particular frame $\{\,\mathbf{e}_a\,\}$. The indeterminacy of propagation of the remaining three

[1] The time derivative of the latter is indeed included in one of the Bianchi field equations, but not in a 'pure' form; because of the dynamical meaning of $\pi_{\alpha\beta}$ discussed below, we regard this as an equation for $E_{\alpha\beta}$.

variables reveals the necessity of tying the description of non-vacuum gravitational phenomena to a thermodynamical description of the matter source fields. This may be achieved, for example, by a phenomenological scheme for dissipative relativistic fluids or a relativistic kinetic theory approach. Second, the combinations of derivatives occurring in the above equations do not have the required symmetric structure. Given our present goal, the target is to choose a suitable matter description and then shuffle the evolution equations for the full set of variables into a FOSH form.

### 2.6 $(1+1+2)$–decomposition

It proves very helpful to consider a $(1+1+2)$–decomposition of all geometrically defined field variables and their dynamical relations. In order to do so, we arbitrarily pick the frame basis field $\mathbf{e}_1$ as a spacelike reference direction. In a small isotropic neighbourhood $\mathcal{U}$ in the local rest 3-space of an arbitrary event $\mathcal{P}$, we establish the convention of regarding those spatial frame components of geometrical objects which contain the index "1" as *longitudinal* with respect to $\mathbf{e}_1$, while regarding those which exclude the index "1" as *transverse* with respect to $\mathbf{e}_1$. Likewise, in $\mathcal{U}$, $\mathbf{e}_1$ shall constitute the *outward* frame derivative while $\mathbf{e}_2$ and $\mathbf{e}_3$ shall be *tangential* frame derivatives. For the frame components of spatial rank-2 symmetric tracefree tensors $a_{\alpha\beta} = a_{\langle\alpha\beta\rangle}$ with squared magnitude $a^2 := \frac{1}{2}(a_{\alpha\beta}a^{\alpha\beta}) \geq 0$, we define a *new* set of frame variables by

$$\begin{array}{lll} a_+ := \frac{1}{2}\,(a_{22}+a_{33}) = -\frac{1}{2}\,a_{11} & a_- := \frac{1}{2\sqrt{3}}\,(a_{22}-a_{33}) & a_\times := \frac{1}{\sqrt{3}}\,a_{23} \\ a_\times := \frac{1}{\sqrt{3}}\,a_{23} & a_2 := \frac{1}{\sqrt{3}}\,a_{31} & a_3 := \frac{1}{\sqrt{3}}\,a_{12}\ , \end{array} \qquad (40)$$

so that

$$a^2 = 3\,(a_+^2 + a_-^2 + a_\times^2 + a_2^2 + a_3^2)\ . \qquad (41)$$

In particular, we apply this to $\{\,\sigma_{\alpha\beta},\, E_{\alpha\beta},\, H_{\alpha\beta}\,\}$. In analogy to Eq. (40), we perform a $(1+1+2)$–decomposition of the spatial commutation functions $n_{\alpha\beta}$ by defining

$$\begin{array}{lll} n := n_{11}+n_{22}+n_{33} & n_+ := -\,n_{11} + \frac{1}{2}\,(n_{22}+n_{33}) & n_- := \frac{1}{2\sqrt{3}}\,(n_{22}-n_{33}) \\ n_\times := \frac{1}{\sqrt{3}}\,n_{23} & n_2 := \frac{1}{\sqrt{3}}\,n_{31} & n_3 := \frac{1}{\sqrt{3}}\,n_{12}\ . \end{array} \qquad (42)$$

The squared magnitude is then given by

$$\frac{1}{2}(n_{\alpha\beta}n^{\alpha\beta}) = \frac{1}{6}\,(n^2 + 2n_+^2) + 3\,(n_-^2 + n_\times^2 + n_2^2 + n_3^2)\ . \qquad (43)$$

Note that only $(n - 2n_+)$, $n_-$ and $n_\times$ transform as *tensor components* under rotations of the spatial frame $\{\,\mathbf{e}_\alpha\,\}$ about the reference $\mathbf{e}_1$-axis.

## 2.7 Derivation of a FOSH evolution system

By applying the commutator (15) to $f = p$ and using Eqs. (38), (39), (36) and (17), we derive an evolution equation for the acceleration $\dot{u}^\alpha$ of the matter fluid tangents $\mathbf{u}$ given by

$$
\begin{aligned}
& \mathbf{e}_0(\dot{u}^\alpha) - c_s^2\,\mathbf{e}_\beta(\frac{1}{3}\,\Theta\,\delta^{\alpha\beta} + \sigma^{\alpha\beta} + \epsilon^{\alpha\beta\gamma}\,\omega_\gamma) \\
= & -[\,c_s^{-2}\,\frac{d^2p}{d\mu^2}\,(\mu+p) - c_s^2 + \frac{1}{3}\,]\,\Theta\,\dot{u}^\alpha - (\dot{u}_\beta + 3c_s^2 a_\beta)\,\sigma^{\alpha\beta} - c_s^2\,n^\alpha{}_\beta\,\omega^\beta \\
& + \epsilon^{\alpha\beta\gamma}\,[\,(2c_s^2-1)\,\omega_\beta\,\dot{u}_\gamma + \Omega_\beta\,\dot{u}_\gamma - c_s^2\,a_\beta\,\omega_\gamma - c_s^2\,n_{\beta\delta}\,\sigma^\delta{}_\gamma\,] \\
& - c_s^2\,(C_1)^\alpha + \Theta\,c_s^{-2}\,\frac{d^2p}{d\mu^2}\,(C_{PF})^\alpha\;. 
\end{aligned}
\tag{44}
$$

Next, contracting the commutator (15), again applied to $f = p$, with $\epsilon^{\alpha\beta\gamma}$ and using Eqs. (38), (36) and (39) leads to the identity

$$
0 = \epsilon^{\alpha\beta\gamma}\,(\mathbf{e}_\beta - a_\beta)\,(\dot{u}_\gamma) - n^\alpha{}_\beta\,\dot{u}^\beta - 2\,c_s^2\,\Theta\,\omega^\alpha \tag{45}
$$

$$
- (\mu+p)^{-1}\,[\,\epsilon^{\alpha\beta\gamma}\,(\mathbf{e}_\beta - a_\beta)\,(C_{PF})_\gamma - n^\alpha{}_\beta\,(C_{PF})^\beta + (c_s^{-2}+1)\,\epsilon^{\alpha\beta\gamma}\,\dot{u}_\beta\,(C_{PF})_\gamma \tag{46}
$$

This identity constitutes the *key step* in achieving FOSH form for the evolution subsystem (30) - (32) and (44) that links the kinematical fluid variables $\Theta$, $\sigma_{\alpha\beta}$ and $\omega^\alpha$ to $\dot{u}^\alpha$ and establishes the sound cone structure on $(\,\mathcal{M},\,\mathbf{g},\,\mathbf{u}\,)$. The trick is to add, on using Eq. (46), $\epsilon^{\alpha\beta\gamma}\,\mathbf{e}_\beta(\dot{u}_\gamma)$ to the left-hand side of Eq. (32), i.e., to change its principle part to the new form $\mathbf{e}_0(\omega^\alpha) + \frac{1}{2}\,\epsilon^{\alpha\beta\gamma}\,\mathbf{e}_\beta(\dot{u}_\gamma)$.

In order to obtain from the extended 1+3 orthonormal frame relations proper partial differential equations such that the FOSH theory applies, we express the coordinate components $e_0{}^\mu := \mathbf{e}_0(x^\mu)$ and $e_\alpha{}^\mu := \mathbf{e}_\alpha(x^\mu)$ of the 1+3 ONF basis $\{\,\mathbf{e}_0,\,\mathbf{e}_\alpha\,\}$ in terms of the comoving local coordinate basis $\{\,\partial_t,\,\partial_i\,\}$ given by

$$
\mathbf{e}_0 := M^{-1}\,\partial_t\;,\quad \mathbf{e}_\alpha := e_\alpha{}^i\,(M_i\,\partial_t + \partial_i)\;; \tag{47}
$$

Here $M = M(t, x^i)$ denotes the *threading lapse function* and $M_i\,dx^i = M_i(t, x^j)\,dx^i$ the dimensionless *threading shift 1-form,* already introduced above in slightly different form. The inverse of the *threading metric* is $h^{ij} := \delta^{\alpha\beta}\,e_\alpha{}^i\,e_\beta{}^j$. The commutator equations (15) and (15) yield

$$
e_\alpha{}^i\,[\,\partial_t M_i + M^{-1}\,(M_i\,\partial_t M + \partial_i M)\,] = \dot{u}_\alpha \tag{48}
$$

$$
M^{-1}\,\partial_t e_\alpha{}^i = -[\,\frac{1}{3}\,\Theta\,\delta_\alpha{}^\beta + \sigma_\alpha{}^\beta + \epsilon_\alpha{}^\beta{}_\gamma\,(\omega^\gamma + \Omega^\gamma)\,](49)^i_\beta
$$

$$
M\,e_{[\alpha}{}^i\,e_{\beta]}{}^j\,(M_i\,\partial_t M_j + \partial_i M_j) = \epsilon_{\alpha\beta\gamma}\,\omega^\gamma \tag{50}
$$

$$
2\,e_{[\alpha}{}^i\,[\,M_i\,\partial_t e_{\beta]}{}^j + \partial_i e_{\beta]}{}^j\,]\,e^\gamma{}_j = 2\,a_{[\alpha}\,\delta^\gamma{}_{\beta]} + \epsilon_{\alpha\beta\delta}\,n^{\delta\gamma}\;, \tag{51}
$$

where $e^\alpha{}_i$ is defined through the relation $e^\alpha{}_i \, e_\alpha{}^j = \delta^j{}_i$. One way of picking the threading lapse function $M$ is provided by parameterising a central integral curve of $\mathbf{u}$, taken as a reference line, by physical *proper time*. Then we have

$$e_0{}^\mu = u^\mu = M_0^{-1} \, \delta^\mu{}_0 \ , \tag{52}$$

where $M_0$ denotes a constant threading lapse function of *unit* magnitude. If, furthermore, the spatial frame $\{ \, \mathbf{e}_\alpha \, \}$ is chosen to be *Fermi-propagated* along $\mathbf{u}$, i.e.,

$$\Omega^\alpha = 0 \ , \tag{53}$$

we obtain reduced evolution equations for $M_i$ and $e_\alpha{}^i$;

$$\partial_t M_i \ = \ \dot{u}_\alpha \, e^\alpha{}_i = \dot{u}_i \tag{54}$$

$$M_0^{-1} \, \partial_t e_\alpha{}^i \ = \ -[ \ \frac{1}{3} \, \Theta \, \delta_\alpha{}^\beta + \sigma_\alpha{}^\beta + \epsilon_\alpha{}^\beta{}_\gamma \, \omega^\gamma \ ] \ e_\beta{}^i \ . \tag{55}$$

Our preparations for the derivation of an evolution system in FOSH form for barotropic perfect fluids from the general $1+3$ ONF equations are now complete. We will use the perfect fluid form of the set of equations (37), (38), (52) - (54), (28) - (34) and (44) with (46), and (36). Of immediate interest for the FOSH structure is only the principle part, the left-hand side in Eq. (1) that describes the dynamical interactions between the various fields. In terms of the frame derivatives $\mathbf{e}_a$ we can represent it by *The frame derivative principle part*:

$$\bar{M}^{AB\,a} \, \mathbf{e}_a(u_B) \ . \tag{56}$$

Using tracefree-adapted irreducible frame variables as defined in Eq. (40) for each of the fluid rate of shear and the electric and magnetic Weyl curvature, a FOSH evolution system can now be obtained, taking certain linear combinations of the equations where necessary, for the following set of 44 dependent dynamical fields: *The dependent geometrical field variables*:

$$u^A = \begin{pmatrix} u_{frame} \\ u_{con} \\ u_{kin,1} \\ u_{kin,2} \\ u_{kin,3} \\ u_{mat} \\ u_{Weyl} \end{pmatrix} , \quad \begin{array}{lcl} u_{frame} & = & [ \ e_\alpha{}^i, \, M_i \ ]^T \\ u_{con} & = & [ \ a^\alpha, \, n_{\alpha\beta} \ ]^T \\ u_{kin,1} & = & [ \ \dot{u}_1, \, \frac{1}{3}(\Theta - 2\sigma_+), \, (\frac{1}{\sqrt{3}}\sigma_3 + \omega_3), \, (\frac{1}{\sqrt{3}}\sigma_2 - \omega_2) \ ]^T \\ u_{kin,2} & = & [ \ \dot{u}_2, \, \frac{1}{3}(\Theta + \sigma_+ + \sqrt{3}\sigma_-), \, (\frac{1}{\sqrt{3}}\sigma_1 + \omega_1), \, (\frac{1}{\sqrt{3}}\sigma_3 - \omega_3) \ ]^T \\ u_{kin,3} & = & [ \ \dot{u}_3, \, \frac{1}{3}(\Theta + \sigma_+ - \sqrt{3}\sigma_-), \, (\frac{1}{\sqrt{3}}\sigma_2 + \omega_2), \, (\frac{1}{\sqrt{3}}\sigma_1 - \omega_1) \ ]^T \\ u_{mat} & = & [ \ \mu \ ] \\ u_{Weyl} & = & [ \ E_+, \, E_-, \, E_1, \, E_2, \, E_3, \, H_+, \, H_-, \, H_1, \, H_2, \, H_3 \ ]^T \end{array} \ .$$

The symmetric $(44 \times 44)$-matrices $\bar{M}^{AB\,a}$ occurring in Eq. (56) are found to assume the forms

*The matrices*:

$$\bar{M}^{AB\,0} = \begin{pmatrix} \mathbf{1}_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \mathbf{1}_9 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \mathbf{K} & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \mathbf{K} & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \mathbf{K} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mathbf{1}_{10} \end{pmatrix}, \tag{57}$$

$$\bar{M}^{AB\,1} = \begin{pmatrix} \mathbf{0}_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \mathbf{0}_9 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \mathbf{B}_1 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \mathbf{B}_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \mathbf{B}_3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mathbf{C}_1 \end{pmatrix}, \tag{58}$$

$$\bar{M}^{AB\,2} = \begin{pmatrix} \mathbf{0}_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \mathbf{0}_9 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \mathbf{B}_3 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \mathbf{B}_1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \mathbf{B}_2 & \cdot & , \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mathbf{C}_2 \end{pmatrix}, \tag{59}$$

$$\bar{M}^{AB\,3} = \begin{pmatrix} \mathbf{0}_{12} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \mathbf{0}_9 & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \mathbf{B}_2 & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \mathbf{B}_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \mathbf{B}_1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \mathbf{C}_3 \end{pmatrix}. \tag{60}$$

$$\mathbf{K} \ : \ = \begin{pmatrix} 1 & \cdot \\ \cdot & c_s^2\,\mathbf{1}_3 \end{pmatrix}, \ \mathbf{B}_1 := -\,c_s^2 \begin{pmatrix} \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}, \tag{61}$$

$$\mathbf{B}_2 \ : \ = -\,c_s^2 \begin{pmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{pmatrix}, \mathbf{B}_3 := -\,c_s^2 \begin{pmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \tag{62}$$

$$
\mathbf{C}_1 := \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1/2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1/2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & -1/2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}, \qquad (63)
$$

$$
\mathbf{C}_2 := \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{3}/2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1/2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & -\sqrt{3}/2 & 1/2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1/2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & -\sqrt{3}/2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1/2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & -1/2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\sqrt{3}/2 & -1/2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1/2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}, \qquad (64)
$$

$$
\mathbf{C}_3 := \begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\sqrt{3}/2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1/2 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1/2 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1/2 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \sqrt{3}/2 & 1/2 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \sqrt{3}/2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1/2 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1/2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & -1/2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
-\sqrt{3}/2 & -1/2 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}. \qquad (65)
$$

The components of all these matrices are just numerical constants since we evaluated the priniple part of our FOSH evolution system in the orthonormal basis $\{\, \mathbf{e}_a \,\}$. When the frame derivatives $\mathbf{e}_a$ in Eq. (56) are substituted for in terms of their coordinate components and the partial derivatives introduced in Eqs. (47) and (52), the non-constant components of the symmetric matrices $M^{AB\,\mu}$ in Eq. (1) with respect to the local coordinate basis $\{\, \partial_\mu \,\}$ can be easily read off.

## 2.8 Characteristics, propagation velocities, and eigenfields

The set of characteristic 3-surfaces $\{\phi = \text{const}\}$ underlying a FOSH evolution system can be interpreted as a collection of wavefronts with phase function $\phi$

across which certain physical quantities may be discontinuous. The associated characteristic eigenfields propagate along bicharacteristic rays *within* these 3-surfaces at velocities $v$, which represent their slopes relative to the direction of $\mathbf{u}$ [3]. To determine the characteristic 3-surfaces, we can assume without loss of generality that at a particular spacetime event the orthonormal frame is oriented in such a way that the orthogonal gradient vector fields, $\xi_a := \nabla_a \phi$, are tangent to a 2-surfaces spanned by $\mathbf{u}$ and $\mathbf{e}_1$. This choice of frame, which then only leaves the freedom of spatial rotations about the $\mathbf{e}_1$-direction, can be made due to the local isotropy of the characteristic cones. Hence, locally the frame axes and the given coordinate lines are made to coincide. In particular, at a point we can take coordinate components of $\mathbf{e}_1$ where $0 = {e_1}^2 = {e_1}^3$ and also $M_i = 0$; however, generically their time derivatives will be non-zero. Thus, we obtain

$$\mathbf{e}_0 = M_0^{-1}\,\partial_t \ , \ \mathbf{e}_1 = {e_1}^1\,\partial_1 \ . \tag{66}$$

### 2.8.1 The characteristic condition

The characteristic condition the vector fields $\xi_\mu$ have to satisfy is

$$0 = \det\,[\,M^{AB\,\mu}\,\xi_\mu\,]. \tag{67}$$

With the choice of frame outlined, their coordinate components are given by $\xi_\mu = -\,v\,u_\mu + {e^1}_\mu$ , where the parameter $v$ coincides with the different possible characteristic propagation velocities of the characteristic eigenfields and ${e^1}_\mu$ denotes the inverse coordinate components of $\mathbf{e}_1$. With Eq. (66) we find, therefore, $\xi_\mu = v\,M_0\,{\delta^0}_\mu + ({e_1}^1)^{-1}\,{\delta^1}_\mu$, leading to

$$\begin{aligned} 0 &= \det\,[\,v\,M_0\,M^{AB\,0} + ({e_1}^1)^{-1}\,M^{AB\,1}\,] \\ &= (M_0)^{44}\,c_s^6\,v^{30}\,(v-c_s)^3\,(v+c_s)^3\,(v-1)^2\,(v+1)^2\,(v-\frac{1}{2})^2\,(v+\frac{1}{2})^2 \end{aligned} \tag{68}$$

Clearly *no* root $v$ of this equation is single-valued, implying a set of characteristic 3-surfaces each of which is degenerate.

### 2.8.2 Characteristic velocities and eigenfields

First, we find 30 characteristic eigenfields of our FOSH evolution system that propagate with velocity $v_1 = 0$ with respect to observers comoving with $\mathbf{u}$; these are

$$u^A_{(1)} = [\ {e_\alpha}^i,\, M_i,\, a^\alpha,\, n_{\alpha\beta},\, \frac{1}{3}(\Theta + \sigma_+ \pm \sqrt{3}\sigma_-), \tag{69}$$

$$(\frac{1}{\sqrt{3}}\sigma_1 \pm \omega_1),\, (\frac{1}{\sqrt{3}}\sigma_2 - \omega_2),\, (\frac{1}{\sqrt{3}}\sigma_3 + \omega_3),\, \mu,\, E_+,\, H_+\ ]^T\ . \tag{70}$$

Second, upon diagonalisation of the principle part in Eq. (1), the set of 6 characteristic eigenfields propagating with velocities $v_{2,3} = \pm\,c_s$ along the sound cone is found to be

$$u^A_{(2)} = [\ \frac{1}{\sqrt{1+c_s^2}}\,(\,c_s\dot{u}_1 \pm \frac{1}{3}(\Theta - 2\sigma_+)\,), \tag{71}$$

$$\frac{1}{\sqrt{1+c_s^2}}\,(\,c_s\dot{u}_2 \pm (\frac{1}{\sqrt{3}}\sigma_3 - \omega_3)\,),\ \frac{1}{\sqrt{1+c_s^2}}\,(\,c_s\dot{u}_3 \pm (\frac{1}{\sqrt{3}}\sigma_2 + \omega_2)\,)\,]^T \tag{72}$$

Here, and in the following, the upper sign applies to outgoing modes and the lower one to incoming modes. By construction, the tracefree-adapted variables of Eq. (40) clearly exhibit the purely *longitudinal* character of the first two eigenfields and the semi-longitudinal of the latter four with respect to the (assumed) spatial propagation direction $\mathbf{e}_1$.

Third, the set of 4 characteristic eigenfields propagating with velocities $v_{4,5} = \pm\,1$ along the null cone is

$$u^A_{(3)} = [\ \frac{1}{\sqrt{2}}\,(\,E_- \mp H_1\,),\ \frac{1}{\sqrt{2}}\,(\,E_1 \pm H_-\,)\ ]^T\ ; \tag{73}$$

each pair corresponds to one of the two possible polarisation states of the freely propagating gravitational field. Again, the tracefree-adapted variables of Eq. (40) nicely reveal that these eigenfields are purely *transverse* to the (assumed) spatial propagation direction $\mathbf{e}_1$.

Finally one obtains 4 characteristic eigenfields propagating with velocities $v_{6,7} = \pm\,\frac{1}{2}$ along timelike 3-surfaces which are

$$u^A_{(4)} = [\ \frac{1}{\sqrt{2}}\,(\,E_3 \mp H_2\,),\ \frac{1}{\sqrt{2}}\,(\,E_2 \pm H_3\,)\ ]^T\ . \tag{74}$$

However these states are not allowed to occur because of the constraint equations: initial data cannopt be set that will lead to these modes being activated [2].

The results of Eqs. (69) - (74) lead to a number of observations. The spacetime metric $\mathbf{g}$, which embodies the local causal structure (and has coordinate components constructed by $g_{\mu\nu} = \eta_{ab}\, e^a{}_\mu\, e^b{}_\nu$), propagates along the timelike reference congruence itself, i.e., with $v = 0$. Parts of the spacetime connection (which contains first derivatives of $\mathbf{g}$) also propagate at $v = 0$, while the remaining parts follow the sound cone. With the present geometrical set of dependent field variables, it is only within the Weyl curvature (which is of second order in the derivatives of $\mathbf{g}$) that one finds modes that propagate *changes* in the state of a gravitational field at the speed of light.

As to the magnitudes of the different propagation velocities it should be noted that, by use of the Bianchi field equations (33) and (34), causal propagation of the gravitational field modes with $|v| \in \{\,0,\,1\,\}$ falls out automatically; no further assumptions are required. To ensure causal propagation of pressure perturbations in the matter fluid, on the other hand, we need to impose the condition $0 \leq c_s < 1$.

It can be easily inferred from the propagation equations along $\mathbf{u}$ for the Weyl curvature divergence equations (23) and (24), that the characteristic velocities relative to $\mathbf{u}$ for the components $(C_4)_2 \mp (C_5)_3$ and $(C_4)_3 \pm (C_5)_2$ are $v = \pm\frac{1}{2}$ too. Hence it follows that the Weyl curvature divergence equations propagate relative to $\mathbf{u}$ at *precisely* the speed that is required to ensure that jump discontinuities in $\mathbf{e}_1(E_3 \mp H_2)$ and $\mathbf{e}_1(E_2 \pm H_3)$ will *always remain physically disallowed* at any instant throughout the dynamical evolution of a cosmological model $(\,\mathcal{M},\,\mathbf{g},\,\mathbf{u}\,)$. It should be emphasised at this stage that this property is completely independent of the presence of matter. That is, of course the jump conditions apply equally to vacuum spacetime configurations.

Jump discontinuities in the outward first frame derivatives of the transverse Weyl curvature characteristic eigenfields $(E_- \mp H_\times)$ and $(E_\times \pm H_-)$ are physically allowed. Clearly, this situation reflects the freedom of specifying four *arbitrary* (non-analytic) real-valued functions $I_{\partial^3 g} := \{\,a_1(x^i),\,a_2(x^i),\,a_3(x^i),\,a_4(x^i)\,\}$ of differentiability class $C^2(\mathcal{U})$ with respect to the zeroth-order derivative level of $\mathbf{g}$ as the initial data for the dynamical degrees of freedom associated with the gravitational field itself.

The key point of all this is that as well as the gravitational wave modes that propagate at the speed of light, there are scalar and vector modes whose characteristics are are timelike (see also [4] for a different approach leading to the same result). Hence the true causal domains in the universe are not necessarily bounded by the light cone; they may be much smaller, depending on the physical conditions holding. We now look at this in the cosmological context.

## 3 The real physical universe: different epochs

Different epochs occur in the history of the universe, with different causal implications. We consider here only the Hot Big Bang era and later times; similar considerations will hold for any inflationary and earlier epochs that might occur before the Hot Big Bang era, but we do not consider them here. Thus the epochs we consider are

**Phase 1**: the *early universe: initially, relativistic*

- spatially homogeneous to high accuracy, speed of sound is $c/\sqrt{3}$, with relativistic diffusion,

Later, *non-relativistic matter, radiation dominated*

- spatially homogeneous to high accuracy, with non-relativistic diffusion and speed of sound.

**Phase 2**: *decoupling and structure formation.*

- structure forms spontaneously because of gravitational attraction after matter radiation decoupling; spatially homogeneous on large averaging scales but with growing inhomogeneities on smaller scales.

**Phase 3**: *late universe: isolated astronomical structures*

- the universe is spatially homogeneous on average (very large scales) but on scales relevant to local physics is very inhomogeneous; isolated clumps of matter such as the Solar system and the Galaxy have separated out of the global expansion and are separated from each other by vast regions of space. Fractional density contrasts are very high (of order $10^{30}$).

### 3.1 Domains of Influence

The true domain of dependence is different in each epoch. As discussed above, timelike characteristics hold for all modes except gravitational radiation, corresponding to the distinction between scalar, vector, and tensor modes in studies of cosmological perturbations. Tensor modes are determined by distant matter, their characteristic velocity is the speed of light, but for pressure-free matter the characteristic velocities of the scalar and vector modes are zero, so they are only directly affected by very close matter.

**Phase 1**: Effective characteristics are those of sound waves; due to the high degree of homogeneity, gravitational waves are negligible.

In the first part, when relativistic matter dominates, the real domain of dependence is a significant part of past light cone: it is bounded by $1/\sqrt{3} = 0.58$ of the light cone, corresponding to perturbations by relativistic sound waves; but even here we do not expect major information propagation at that speed. In the second part, when non-relativistic matter dominates, the real domain of dependence is a few percent of the past light cone because the speed of sound is non-relativistic.

**Phase 2:** Characteristics are timelike curves and. the real domain of dependence is very local.

Only pressure-free scalar and vector modes are significant in linear phase of growth. There will be large scale matter flows but at sub-relativistic velocities. The domain of dependence for each inhomogeneity will be the 'catchment volume' from which it succeeds in accreting material. Again the real domain of dependence is a few percent of the past light cone because these speeds are non-relativistic.

**Phase 3**: Just a small world tube around our world line is significant (out to Sun)

hence the real domains of dependence in cosmology are much smaller than implied by the light cone, and this can

¿From this, one can draw causal diagrams showing the different epochs and their true domains of causality: the real physical domain of dependence. They are quite different than the standard causal horizons determined by the past light cone, because scalar modes dominate all these epochs and to a lesser extent vector modes, with tensor modes being negligible in all cases. Of course the past light cone is crucial to astronomical and cosmological observations, but the relevant photons from distant domains have very little physical effect on earth; indeed we have to develop exstremely sensitive detectors in order to record them. For them the light cone and its associated limits (the visual horizon) are relevant.

## 4 Physical effects in the real universe

Finally we ask if there are ever times in the history of the universe when gravitational wave effects are significant, as opposed to time-dependent tidal forces.From the phase plane analysis of the eraly universe (G0 models) reported by John Wainwright, we arrive at a series of tentative conclusions [8]:

- *Strong gravity will occur in the early universe, associated with local restrictions on causality.*
- *Gravitational waves are not important in the cosmological context, but tidal forces are and indeed are often more important than the gravitational fields caused directly by the matter.*
- *The relation between tidal forces and vorticity is unclear and may contain some of the most interesting physics.*

The relation between them is that - if our conjectures are correct - in the early universe, energy and information mainly propagate along timelike world lines rather than on null rays. When matter moves relative to the irrotational reference congruence, as must be the case when vorticity is important, then the energy and information will flow with the matter. The primary effect of the gravitational field is in determining the motion of matter through Coulomb-like effects; on the other hand, the effect of the matter on the gravitational field is primarily through concentrating that field into small regions, while conserving the constraints which embody the Gauss law underlying the Coulomb-like behaviour. The effect of spatial curvature is to generate oscillatory behaviour in tidal forces as this concentration takes place, as seems to be characteristic of generic cosmological singularities; but this is not wavelike in the sense of conveying information to different regions, it is just a localised oscillation.

If this is correct, we have a major issue to consider: are there any epochs in the evolution of realistic universe models where tensor (gravitational wave) modes are important? This will be distinguished from the Bianchi Type IX oscillations which are 'silent universes' characterised by ODE's. Could there be good universe models where such genuinely GR dynamical effects are important? One issue here is what characterises a gravitational waves; I suggest that the criterion is that both $curlE \neq 0$ and $curlH \neq 0$ so that in the Maxwell-like equations for the Weyl tensor, we non-trivial coupling leading to a (second-order) wave equation on taking the tiome derivative of either propagation equation. It is issues such as these that need investigation in further development of themes studied here.

## 4.1 Local dynamics versus non-local

At the present time gravitational wave effects are insignificant: and that is why a local Newtonian approximation, and indeed the existence of isolated systems, is possible. However this holds only once one has factored out the background geometry and dynamics, which are of course determined via teh field equations and boundary conditions. It is precisely that full effect (incorporating integrals over all matter) that is the concern of cosmology, not just the perturbation effects around the background geometry. The local evolution allowed in the solar system is possible because we are situated in an almost FLRW background unievrse; there are other models for which this is not true at all times, e.g. those that are like a Bianchi I model at early times (and that will be like a Bianchi VII model at late times). In them GR effects (based in the effect sof spatial anisoropy, and perhaps gravitational waves in some cases) were important at early tmes and may be important at late times but are not so now.

We may paraphrase; the universe does not interfere with local physics but does influence it. It allows isolated systems to exist. The issue is, how large is the set of space-times in which this may be true ? How large is the set where anisotropic GR effects are never important?

# 5 Appendix: The arrow of time

A particularly important kind of top-down action from the cosmos to complex systems is the causal link that governs the choice of the local arrow of time every where in the universe, as discussed in the previous section. It seems likely, for example, that the cosmos puts boundary conditions on solutions to Maxwell's equations that determine the local electro-magnetic arrow of time at the micro-physical level , which in turn determines this arrow at the macro level. How this all happens is a crucial question.

I will here briefly relate the epochs discussed above to the issue of the arrow of time and how that problem looks different at each of these epochs.

### 5.0.1 Relations

In **Phase 1**: the expansion of the universe provides a direction of time for quasi-equilibrium physics,

- the falling temperature gives a unique direction to it all, controls statistical processes linking matter and radiation

- these are mainly reversible because mainly dominated by equilibrium processes, so no arrow of time entailed

- the non-equilibrium epochs give the real arrow of time, because their remnants would not be there otherwise

e.g. neutrons leading to helium 4, the underlying non-equilibrium process is the decay of free neutrons [binary collisions can't keep equilibrium because electrons/positrons are gone; the reverse of neutron decay is a 3-body interaction that will be improbable.Thus the one-body decay to 3 remnants is time preferred over the reverse three body collision]

Thus in this epoch, the expansion marks out the arrow of time uniquely

Would the arrow of time be opposite if the universe contracted? - for equilibrium processes, yes: not for non-equilibrium. Constituents would be different and in a collapse phase there would be different matter constituents at each temperature,

In **Phase 2**, the universe starts off with smooth initial conditions and ends up clumpy through structure formation due to gravitional attraction..

The time reverse is possible but does not happen because of smooth initial conditions.. These improbable initial conditions are supposed to be explained by inflation, but that is disputed: Indeed R Penrose claims the opposite is true: inflation cannot explain the smooth initial conditions needed to give the macro arrow of time because of (i) quantum foam and (ii) black holes.

Either the entropy law does not apply to gravitational structure formation, or the consequences are the opposite of what is usually stated.

Maybe it applies but with such different conditions (dynamics: long range gravitational attraction with no negative masses; initial conditions: very smooth ) that the outcome is the opposite of what is usually stated. which is the case.? If so then the increase of entropy is not a fundamental law but is a consnequence of special initial conditions NB: We have no good definition of gravitational entropy in terms of coarse graining or phase space volumes. A major mssing item in classical GR.

In **Phase 3**, the arrow of time in a local isolated system comes either

- from influences crossing a spatial separating boundary, expressed in associated boundary conditions (e.g. an 'outgoing radiation condition', which is really a no-incoming radiation condition), or

- through initial conditions at the start in the local region, perhaps compared with those at the end (R Penrose), setting up an arrow of time initially, then preserved by conservation laws

- or through extra selection condition, imposed afresh at each time independent of earlier times (O Penrose and I Percival).

This is quite different than in the other two cases. The expansion of the universe would seem to have no appreciable impact. Itis possibel that the source of the arrow is the electromagnetic arrow based in differences in the far distant future and past; but how does that influence local mechanical and chemical events? And how does that distant influence get conveyed to a local system across a bounding sphere surrounding the system?

#### 5.0.2 Issues:

** how does each mechanism by itself relate to the arrow(s) of time?

** how do the different mechanisms relate to each other (giving the same time direction)?

** how would they behave if there were conditions leading to a reversal of the arrow of time?

# References


[1] H van Elst and G F R Ellis.Phys Rev D59 (1999), 024013, 1-22

[2] Henk van Elst, George F R Ellis , Bernd G Schmidt. Phys.Rev. D62 (2000) 104023 [gr-qc/0007003]

[3] R. Courant and D. Hilbert, *Methods of Mathematical Physics* vol. 2 (Interscience Publishers, New York, 1962).

[4] J. Ehlers, A. R. Prasanna and R. A. Breuer, Propagation of gravitational waves through pressureless matter, Class. Quantum Grav. **4**, 253 (1987).

[5] H. van Elst and C. Uggla, General relativistic $1 + 3$ orthonormal frame approach, Class. Quantum Grav. **14**, 2673 (1997).

[6] H. Friedrich, Hyperbolic reductions for Einstein's equations, Class. Quantum Grav. **13**, 1451 (1996).

[7] K. O. Friedrichs, Symmetric hyperbolic linear differential equations, Commun. Pure Appl. Math. **7**, 345 (1954).

[8] Uggla C, van Elst H, Wainwright J and Ellis G F R 2003 The past attractor in inhomogeneous cosmology *Archive* gr-qc/0304002