Combinatorial invariants on finite graphs and the syzygy theory of monomial ideals

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Abstract.

The research area Combinatorics and Commutative Algebra was born in the middle of 1970s out of Stanley [6] and Reisner [5] under the great influence of Hochster [4]. It's been 50 years since then, nowadays, quite a lot of research papers have been published in this field. Stanley [6] introduced methods from commutative algebra into combinatorics, while classical results in combinatorics have exerted profound influence on the development of commutative algebra [3]. Our aim is to demonstrate how combinatorial invariants on finite graphs can be translated via the syzygy theory of monomials, thereby deriving new results from known findings.

Let G be a finite graph on $[n] = \{1, \ldots, n\}$ and $E(G)$ the set of edges of G. A *vertex cover* of G is a subset $C \subset [n]$ for which $e \cap C \neq \emptyset$ for all $e \in E(G)$. Let $\tau_{\max}(G)$ denote the maximum size of minimal vertex covers of G. In [1], it is shown that $\tau_{\max}(G) \geq \lceil 2\sqrt{n} - 2 \rceil$. Let $\varkappa(G)$ denote the *vertex connectivity* of G. In other words, $\varkappa(G)$ is the minimum cardinality of a subset $W \subset [n]$ for which the induced subgraph $G|_{[n]\setminus W}$ is disconnected. In combinatorics on finite graphs, the invariants $\varkappa(G)$ and $\tau_{\max}(G)$ appear to be fundamentally distinct. However, in certain restricted settings, the syzygy theory of monomial ideals in commutative algebra establishes an explicit relationship between these invariants.

This is a joint work [2] with Tài Hà.

References

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