PHD QUALIFYING EXAM IN TOPOLOGY

1. Let X, Y and Z be topological spaces and $f : X \to Y$ and $g : Y \to Z$ continuous maps.

(a) Define the group $C_n(X)$ of singular n–chains in X
(b) Define the homomorphism $f_\# : C_n(X) \to C_n(Y)$
(c) Show that $(g \circ f)_\# = g_\# \circ f_\#$
(d) Define the group $C^n(X)$ of singular n–cochains in X
(e) Define the homomorphism $f^\# : C^n(Y) \to C^n(X)$

2. The CW–complex X is obtained from two copies of the Möbius band by identifying their (oriented) boundary circles via a map f of degree n.

(a) Calculate the fundamental group $\pi_1(X, x)$;
(b) Calculate the homology groups $H_*(X; \mathbb{Z})$.

3. Show that any closed n-manifold X such that $H_*(X; \mathbb{Z}/2) = H_*(S^n; \mathbb{Z}/2)$ is orientable.

4. Is the one-point union $S^2 \vee S^4$ homotopy equivalent to a closed manifold ? Explain why or why not.

5. Consider the double covering $p : S^2 \to \mathbb{R}P^2$.

(a) Prove that, for any abelian group A, the induced homomorphisms $p^* : \tilde{H}^*(\mathbb{R}P^2; A) \to \tilde{H}^*(S^2; A)$ are all zero.
(b) Prove that the map p is not homotopic to a constant map.

6. For any $n \geq 1$, calculate the set $[\mathbb{C}P^n, \mathbb{C}P^n]$ of the homotopy classes of continuous maps from $\mathbb{C}P^n$ to itself.