PHD QUALIFYING EXAM IN TOPOLOGY

1. Let $f : (X, A) \to (Y, B)$ be a continuous map of pairs of (arbitrary) topological spaces such that $f(X) \subset B$. Prove that the homomorphism $f_* : H_*(X, A) \to H_*(Y, B)$ induced by f in singular homology is zero.

2. The CW–complex X is obtained by attaching a copy of the Möbius band to the boundary of a punctured torus $T^2 \setminus \operatorname{Int} D^2$ using a homeomorphism of their boundary circles.

(a) Calculate the fundamental group $\pi_1(X)$;

(b) Calculate the homology groups $H_*(X; \mathbb{Z})$.

3. Let X be a finite CW–complex, and assume that X is connected.

(a) Prove that $H_1(X; \mathbb{Z}) = 0$ if and only if $H_1(X; \mathbb{Z}/p) = 0$ for all prime numbers $p \geq 2$.

(b) Will the same statement be true if H_1 is replaced by H^1 ?

4. Calculate the cohomology ring $H^*(\mathbb{C}\mathrm{P}^3; \mathbb{Z})$ and use it to prove that $S^2 \times S^4$ and $\mathbb{C}\mathrm{P}^3$ are not homotopy equivalent.

5. Denote by M_g a closed oriented connected surface of genus g. Prove that there is no covering map $p : M_g \to M_h$ such that $g < h$.

6. Calculate the homotopy groups $\pi_k(\mathbb{C}\mathrm{P}^n)$ for all $n \geq 1$ and $k \leq 2n + 1$.