PHD QUALIFYING EXAM IN TOPOLOGY

1. Compute the fundamental group and homology groups of the topological space obtained by removing the three coordinate axes from $\mathbb{R}^3$.

2. Let X be a topological space, and let $A \subset X$ be its retract. Prove that $H_*(X) \cong H_*(A) \oplus H_*(X, A)$.

3. The topological space X is obtained by attaching a Möbius band M to the projective plane $\mathbb{R}\mathrm{P}^2$ via a homeomorphism of the boundary circle of M to the standard $\mathbb{R}\mathrm{P}^1$ in $\mathbb{R}\mathrm{P}^2$.

(a) Compute $\pi_1(X)$.

(b) Compute $H_*(X)$ and $H^*(X)$.

(c) Compute $H_*(X; \mathbb{Z}/2)$ and $H^*(X; \mathbb{Z}/2)$.

4. Let X be a closed connected orientable 4-manifold whose second homology group $H_2(X)$ has rank one. Prove that there does not exist a free action of the group $\mathbb{Z}/2$ on X.

5. Let $f : X \to Y$ be a continuous map between two closed connected oriented n–manifolds such that $\deg f = 1$. Prove that the induced homomorphism $f_* : \pi_1(X) \to \pi_1(Y)$ is surjective.

6. Let Σ_g be a closed connected orientable surface of genus $g \geq 1$.

(a) Prove that $\pi_k(\Sigma_g) = 0$ for all $k \geq 2$.

(b) Use part (a) to prove that $\pi_1(\Sigma_g)$ is not isomorphic to a free group.