Qualifying Exam - Algebraic Topology
Sept 1, 2021

Do all six problems
You may use standard formulae for computations.

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(1) (a) Show that $\mathbb{R}P^3$ and $\mathbb{R}P^2 \vee S^3$ have isomorphic homology groups in all degrees.
(b) Show that they are not homotopy equivalent in two ways:
(i) by considering the homology groups of their universal covers
(ii) by considering their cohomology rings.

(2) (a) Compute the fundamental groups of $\mathbb{R}P^2 \vee \mathbb{R}P^2$ and $\mathbb{R}P^2 \times \mathbb{R}P^2$.
(b) Show that $\mathbb{R}P^2 \vee \mathbb{R}P^2$ is not a retract of $\mathbb{R}P^2 \times \mathbb{R}P^2$.
(c) Show that any map $f\colon \mathbb{R}P^2 \vee \mathbb{R}P^2 \to S^1$ is null-homotopic.

(3) Let F be the complement of two open disks in the projective plane $\mathbb{R}P^2$. Determine all connected surfaces that arise as 3–fold covers of F.

(4) Let Y be a CW complex where

$$H_i(Y;\mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2, \text{ or } 7 \\ \mathbb{Z}/6\mathbb{Z} & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

Calculate the cohomology groups $H^*(Y;\mathbb{Z})$ and $H^*(Y;\mathbb{Z}/3\mathbb{Z})$.

(5) Let M be an orientable n–manifold. Suppose N is a closed, connected $(n-1)$–submanifold that separates M. Prove that N is orientable.

(6) Let $Y = S^1 \times S^1 \times S^1$.
(a) Compute $H_2(Y;\mathbb{Z})$.
(b) Compute $\pi_2(Y)$.

$Hom(A,B) =$

$A\backslash B$	$\mathbb{Z}$	$\mathbb{Z}/m$
$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}/m$
$\mathbb{Z}/n$	0	$\mathbb{Z}/\gcd(m,n)$

$Ext(A,B) =$

$A\backslash B$	$\mathbb{Z}$	$\mathbb{Z}/m$
$\mathbb{Z}$	0	0
$\mathbb{Z}/n$	$\mathbb{Z}/n$	$\mathbb{Z}/\gcd(m,n)$

$Tor(A,B) =$

$A\backslash B$	$\mathbb{Z}$	$\mathbb{Z}/m$
$\mathbb{Z}$	0	0
$\mathbb{Z}/n$	0	$\mathbb{Z}/\gcd(m,n)$